Section 2.6 problems:

Problems 1-8: Please see the answers to the odd numbered problems at the end of the book. For the even numbers here are the answers:

2. We can write the equation in the form

$$\frac{dx}{dt} = \frac{x^2 - t^2}{2tx} = \frac{1}{2}\left(\frac{x}{t} - \frac{t}{x}\right),$$

which shows that it is homogeneous. At the same time, it is a Bernoulli equation because it can be written as

$$\frac{dx}{dt} - \frac{1}{2t}x = -\frac{t}{2}x^{-1},$$

4. This is a Bernoulli equation.

6. Dividing this equation by $\theta d\theta$, we obtain

$$\frac{dy}{d\theta} - \frac{1}{\theta}y = \frac{1}{\sqrt{\theta}}y^{1/2}.$$

Therefore, it is a Bernoulli equation. It can also be written in the form

$$\frac{dy}{d\theta} = \frac{y}{\theta} + \sqrt{\frac{y}{\theta}},$$

and so it is homogeneous too.

8. We can rewrite the equation in the form

$$\frac{dy}{dx} = \frac{\sin(x+y)}{\cos(x+y)} = \tan(x+y).$$

Thus, it is of the form $dy/dx = G(ax + by)$ with $G(t) = \tan t$.

Problem 12 solution:

From

$$\frac{dy}{dx} = -\frac{x^2+y^2}{2xy} = -\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right),$$

making the substitution $v = y/x$, we obtain

$$v + x\frac{dv}{dx} = -\frac{1}{2}\left(\frac{1}{v}+v\right) = -\frac{1+v^2}{2v} \quad \Rightarrow \quad x\frac{dv}{dx} = -\frac{1+v^2}{2v} - v = -\frac{1+3v^2}{2v}$$

$$\Rightarrow \quad \frac{2v\,dv}{1+3v^2} = -\frac{dx}{x} \quad \Rightarrow \quad \int \frac{2v\,dv}{1+3v^2} = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{3}\ln\left(1+3v^2\right) = -\ln|x| + C_2 \quad \Rightarrow \quad 1+3v^2 = C_1|x|^{-3},$$

where $C_1 = e^{3C_2}$ is any positive constant. Making the back substitution, we finally get

$$1+3\left(\frac{y}{x}\right)^2 = \frac{C_1}{|x|^3} \quad \Rightarrow \quad 3\left(\frac{y}{x}\right)^2 = \frac{C_1}{|x|^3} - 1 = \frac{C_1-|x|^3}{|x|^3}$$

$$\Rightarrow \quad 3|x|y^2 = C_1 - |x|^3 \quad \Rightarrow \quad 3|x|y^2 + |x|^3 = C_1 \quad \Rightarrow \quad 3xy^2 + x^3 = C\,,$$

where $C = \pm C_1$ is any nonzero constant.

Problem 20 solution:

Substitution $z = x - y$ yields

$$1 - \frac{dz}{dx} = \sin z \quad \Rightarrow \quad \frac{dz}{dx} = 1 - \sin z \quad \Rightarrow \quad \frac{dz}{1-\sin z} = dx$$

$$\Rightarrow \quad \int \frac{dz}{1-\sin z} = \int dx = x + C.$$

The left-hand side integral can be found as follows.

$$\int \frac{dz}{1-\sin z} = \int \frac{(1+\sin z)dz}{1-\sin^2 z} = \int \frac{(1+\sin z)dz}{\cos^2 z}$$

$$= \int \sec^2 z + \int \tan z \sec z\, dz = \tan z + \sec z.$$

Thus, a general solution is given implicitly by

$$\tan(x-y) + \sec(x-y) = x + C.$$

Problem 24 solution:

We divide this Bernoulli equation by $y^{1/2}$ and make a substitution $v = y^{1/2}$.

$$y^{-1/2}\frac{dy}{dx} + \frac{1}{x-2}y^{1/2} = 5(x-2)$$
$$\Rightarrow \qquad 2\frac{dv}{dx} + \frac{1}{x-2}v = 5(x-2) \qquad \Rightarrow \qquad \frac{dv}{dx} + \frac{1}{2(x-2)}v = \frac{5(x-2)}{2}.$$

An integrating factor for this linear equation is

$$\mu(x) = \exp\left[\int \frac{dx}{2(x-2)}\right] = \sqrt{|x-2|}.$$

Therefore,

$$\begin{aligned} v(x) &= \frac{1}{\sqrt{|x-2|}} \int \frac{5(x-2)\sqrt{|x-2|}}{2}\,dx \\ &= \frac{1}{\sqrt{|x-2|}}\left(|x-2|^{5/2} + C\right) = (x-2)^2 + C|x-2|^{-1/2}. \end{aligned}$$

Since $y = v^2$, we finally get

$$y = \left[(x-2)^2 + C|x-2|^{-1/2}\right]^2.$$

In addition, $y \equiv 0$ is a (lost) solution.

Problem 30 solution:

We make a substitution

$$x = u + h, \quad y = v + k,$$

where h and k satisfy the system (14) in the text, i.e.,

$$\begin{cases} h + k - 1 = 0 \\ k - h - 5 = 0. \end{cases}$$

Solving yields $h = -2$, $k = 3$. Thus, $x = u - 2$ and $y = v + 3$. Since $dx = du$, $dy = dv$, this substitution leads to the equation

$$(u+v)du + (v-u)dv = 0 \qquad \Rightarrow \qquad \frac{dv}{du} = \frac{u+v}{u-v} = \frac{1+(v/u)}{1-(v/u)}.$$

This is a homogeneous equation, and a substitution $z = v/u$ $(v' = z + uz')$ yields

$$z + u\frac{dz}{du} = \frac{1+z}{1-z} \qquad \Rightarrow \qquad u\frac{dz}{du} = \frac{1+z}{1-z} - z = \frac{1+z^2}{1-z}$$

$$\Rightarrow \qquad \frac{(1-z)dz}{1+z^2} = \frac{du}{u}$$

$$\Rightarrow \qquad \arctan z - \frac{1}{2}\ln\left(1+z^2\right) = \ln|u| + C_1$$

$$\Rightarrow \qquad 2\arctan\frac{v}{u} - \ln\left[u^2\left(1+z^2\right)\right] = 2C_1$$

$$\Rightarrow \qquad 2\arctan\frac{v}{u} - \ln\left(u^2+v^2\right) = C.$$

The back substitution yields

$$2\arctan\left(\frac{y-3}{x+2}\right) - \ln\left[(x+2)^2 + (y-3)^2\right] = C.$$