Section 4.3 problems:

Problem 16 solution:

First, we find the roots of the auxiliary equation.

$$r^2 - 3r - 11 = 0 \qquad \Rightarrow \qquad r = \frac{3 \pm \sqrt{3^2 - 4(1)(-11)}}{2} = \frac{3 \pm \sqrt{53}}{2}\,.$$

These are real distinct roots. Hence, a general solution to the given equation is

$$y(t) = c_1 e^{(3+\sqrt{53})t/2} + c_2 e^{(3-\sqrt{53})t/2}.$$

Problem 24 solution:

The auxiliary equation for this problem is $r^2 + 9 = 0$. The roots of this equation are $r = \pm 3i$, and a general solution is given by $y(t) = c_1 \cos 3t + c_2 \sin 3t$, where c_1 and c_2 are arbitrary constants. To find the solution that satisfies the initial conditions, $y(0) = 1$ and $y'(0) = 1$, we solve a system

$$y(0) = (c_1 \cos 3t + c_2 \sin 3t)\,|_{t=0} = c_1 = 1$$
$$y'(0) = (-3c_1 \sin 3t + 3c_2 \cos 3t)\,|_{t=0} = 3c_2 = 1\,.$$

Solving this system of equations yields $c_1 = 1$ and $c_2 = 1/3$. Thus

$$y(t) = \cos 3t + \frac{\sin 3t}{3}$$

is the desired solution.

Section 6.2 problems (only the answers are given).

Problem 12 solution:

$$c_1e^x + c_2e^{-3x} + c_3xe^{-3x}$$

Problem 20 solution:

$$e^{-x} - e^{-2x} + e^{-4x}$$

Section 4.4 problems:

Problem 16 solution:

The corresponding homogeneous equation has the auxiliary equation $r^2 - 1 = 0$, whose roots are $r = \pm 1$. Thus, in the expression $\theta_p(t) = (A_1 t + A_0)\cos t + (B_1 t + B_0)\sin t$ none of the terms is a solution to the homogeneous equation. We find

$$\theta_p(t) = (A_1 t + A_0)\cos t + (B_1 t + B_0)\sin t$$

$$\Rightarrow \qquad \theta_p'(t) = A_1 \cos t - (A_1 t + A_0)\sin t + B_1 \sin t + (B_1 t + B_0)\cos t$$

$$= (B_1 t + A_1 + B_0)\cos t + (-A_1 t - A_0 + B_1)\sin t$$

$$\Rightarrow \qquad \theta_p''(t) = B_1 \cos t - (B_1 t + B_0 + A_1)\sin t - A_1 \sin t + (-A_1 t - A_0 + B_1)\cos t$$

$$= (-A_1 t - A_0 + B_1)\cos t + (-B_1 t - B_0 - 2A_1)\sin t.$$

Substituting these expressions into the original differential equation, we get

$$\theta_p'' - \theta_p = (-A_1 t - A_0 + 2B_1)\cos t + (-B_1 t - B_0 - 2A_1)\sin t$$

$$- (A_1 t + A_0)\cos t - (B_1 t + B_0)\sin t$$

$$= -2A_1 t \cos t + (-2A_0 + 2B_1)\cos t - 2B_1 t \sin t + (-2A_1 - 2B_0)\sin t = t \sin t.$$

Equating the coefficients, we see that

$$\begin{aligned} -2A_1 &= 0 \\ -2A_0 + 2B_1 &= 0 \\ -2B_1 &= 1 \\ -2A_1 - 2B_0 &= 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A_1 &= 0 \\ A_0 = B_1 &= -1/2 \\ B_1 &= -1/2 \\ B_0 = -A_1 &= 0. \end{aligned}$$

Therefore, a particular solution of the nonhomogeneous equation $\theta'' - \theta = t \sin t$ is given by $\theta_p(t) = -(t \sin t + \cos t)/2$.

Problem 22 solution:

The nonhomogeneous term of the original equation is $24t^2e^t$. Therefore, a particular solution has the form $x_p(t) = t^s\left(A_2t^2 + A_1t + A_0\right)e^t$. The corresponding homogeneous differential equation has the auxiliary equation $r^2 - 2r + 1 = (r-1)^2 = 0$. Since $r = 1$ is its double root, s is chosen to be 2, and a particular solution to the nonhomogeneous equation has the form

$$x_p(t) = t^2\left(A_2t^2 + A_1t + A_0\right)e^t = \left(A_2t^4 + A_1t3 + A_0t^2\right)e^t.$$

We compute

$$\begin{aligned} x_p' &= \left[A_2t^4 + (4A_2 + A_1)\,t^3 + (3A_1 + A_0)\,t^2 + 2A_0t\right]e^t, \\ x_p'' &= \left[A_2t^4 + (8A_2 + A_1)\,t^3 + (12A_2 + 6A_1 + A_0)\,t^2 + (6A_1 + 4A_0)\,t + 2A_0\right]e^t. \end{aligned}$$

Substituting these expressions into the original differential equation yields

$$x_p'' - 2x_p' + x_p = \left[12A_2t^2 + 6A_1t + 2A_0\right]e^t = 24t^2e^t.$$

Equating coefficients yields $A_1 = A_0 = 0$ and $A_2 = 2$. Therefore, $x_p(t) = 2t^4e^t$.

Problem 32 solution:

From the form of the right-hand side, we conclude that a particular solution should be of the form

$$y_p(t) = t^s\left(A_6t^6 + A_5t^5 + A_4t^4 + A_3t^3 + A_2t^2 + A_1t + A_0\right)e^{-3t}.$$

Since $r = -3$ is a simple root of the auxiliary equation, we take $s = 1$. Therefore,

$$y_p(t) = \left(A_6t^7 + A_5t^6 + A_4t^5 + A_3t^4 + A_2t^3 + A_1t^2 + A_0t\right)e^{-3t}.$$

Section 4.5 problems:

Problem 28 solution:

The roots of the auxiliary equation, $r^2 + r - 12 = 0$, are $r = -4$ and $r = 3$. This gives a general solution to the corresponding homogeneous equation of the form $y_h(t) = c_1 e^{-4t} + c_2 e^{3t}$. We use the superposition principle to find a particular solution to the given nonhomogeneous equation.

$$y_p = A_0 e^t + B_0 e^{2t} + C_0 \quad \Rightarrow \quad y_p' = A_0 e^t + 2B_0 e^{2t} \quad \Rightarrow \quad y_p'' = A_0 e^t + 4B_0 e^{2t};$$
$$y_p'' + y_p' - 12 y_p = -10 A_0 e^t - 6 B_0 e^{2t} - 12 C_0 = e^t + e^{2t} - 1.$$

Therefore, $A_0 = -1/10$, $B_0 = -1/6$, $C_0 = 1/12$, and a general solution to the original equation is

$$y(t) = -\frac{e^t}{10} - \frac{e^{2t}}{6} + \frac{1}{12} + c_1 e^{-4t} + c_2 e^{3t}.$$

Next, we find c_1 and c_2 such that the initial conditions are satisfied. Since

$$y'(t) = -\frac{e^t}{10} - \frac{e^{2t}}{3} - 4c_1 e^{-4t} + 3c_2 e^{3t},$$

we have

$$\begin{aligned} 1 = y(0) &= -1/10 - 1/6 + 1/12 + c_1 + c_2 \\ 3 = y'(0) &= -1/10 - 1/3 - 4c_1 + 3c_2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 + c_2 &= 71/60 \\ -4c_1 + 3c_2 &= 103/30. \end{aligned}$$

Solving yields $c_1 = 1/60$, $c_2 = 7/6$. With these constants, the solution becomes

$$y(t) = -\frac{e^t}{10} - \frac{e^{2t}}{6} + \frac{1}{12} + \frac{e^{-4t}}{60} + \frac{7e^{3t}}{6}.$$