Section 7.2 problems.

Problem 4 solution:

For $s > 3$, we have

$$\begin{aligned}\mathcal{L}\left\{te^{3t}\right\}(s) &= \int_0^\infty e^{-st}te^{3t}\,dt = \int_0^\infty te^{(3-s)t}\,dt = \lim_{N\to\infty}\int_0^N te^{(3-s)t}\,dt \\ &= \lim_{N\to\infty}\left(\frac{t}{3-s}-\frac{1}{(3-s)^2}\right)e^{(3-s)t}\Big|_0^N \\ &= \lim_{N\to\infty}\left[\frac{1}{(3-s)^2}+\left(\frac{N}{3-s}-\frac{1}{(3-s)^2}\right)e^{(3-s)N}\right] = \frac{1}{(s-3)^2}\,.\end{aligned}$$

Problem 12 solution:

Splitting the integral in the definition of Laplace transform, we get

$$\begin{aligned}\mathcal{L}\{f(t)\}(s) &= \int_0^\infty e^{-st}f(t)\,dt = \int_0^3 e^{-st}e^{2t}\,dt + \int_3^\infty e^{-st}\cdot 1\,dt \\ &= \frac{e^{(2-s)t}}{2-s}\Big|_0^3 - \frac{e^{-st}}{s}\Big|_3^\infty = \frac{1-e^{-3(s-2)}}{s-2}+\frac{e^{-3s}}{s}\,,\end{aligned}$$

which is valid for all $s > 2$.

Problem 16 solution:

Using the linearity of Laplace transform and Table 7.1 in the text, we get

$$\begin{aligned}\mathcal{L}\left\{t^2-3t-2e^{-t}\sin 3t\right\}(s) &= \mathcal{L}\left\{t^2\right\}(s) - 3\mathcal{L}\{t\}(s) - 2\mathcal{L}\left\{e^{-t}\sin 3t\right\}(s) \\ &= \frac{2!}{s^{2+1}} - 3\,\frac{1!}{s^{1+1}} - 2\,\frac{3}{(s+1)^2+3^2} \\ &= \frac{2}{s^3}-\frac{3}{s^2}-\frac{6}{(s+1)^2+9}\,,\end{aligned}$$

valid for $s > 0$.

Problem 22 solution:

Since the function $g_1(t) \equiv 0$ is continuous on $(-\infty, \infty)$ and $f(t) = g_1(t)$ for t in $[0, 2)$, we conclude that $f(t)$ is continuous on $[0, 2)$ and continuous from the left at $t = 1$. The function $g_2(t) \equiv t$ is also continuous on $(-\infty, \infty)$, and so $f(t)$ (which is the same as $g_2(t)$ on $[2, 10]$) is continuous on $(2, 10]$. Since

$$\lim_{t \to 2^-} f(t) = 0 \neq 2 \lim_{t \to 2^+} f(t),$$

$f(t)$ has a jump discontinuity at $t = 2$. Thus $f(t)$ is piecewise continuous on $[0, 10]$. The graph of $f(t)$ is depicted in Fig. **7–A** on page 263.

Section 7.3 problems.

Problem 8 solution:

Since $(1 + e^{-t})^2 = 1 + 2e^{-t} + e^{-2t}$, we have from the linearity of the Laplace transform that

$$\mathcal{L}\left\{(1 + e^{-t})^2\right\}(s) = \mathcal{L}\{1\}(s) + 2\mathcal{L}\left\{e^{-t}\right\}(s) + \mathcal{L}\left\{e^{-2t}\right\}(s).$$

From Table 7.1 of the text, we get

$$\mathcal{L}\{1\}(s) = \frac{1}{s}, \quad \mathcal{L}\left\{e^{-t}\right\}(s) = \frac{1}{s+1}, \quad \mathcal{L}\left\{e^{-2t}\right\}(s) = \frac{1}{s+2}.$$

Thus

$$\mathcal{L}\left\{(1 + e^{-t})^2\right\}(s) = \frac{1}{s} + \frac{2}{s+1} + \frac{1}{s+2}, \quad s > 0.$$

Problem 10 solution:

Since

$$\mathcal{L}\left\{e^{2t} \cos 5t\right\}(s) = \frac{s-2}{(s-2)^2 + 25},$$

we use Theorem 6 to get

$$\mathcal{L}\left\{te^{2t} \cos 5t\right\}(s) = \mathcal{L}\left\{t\left(e^{2t} \cos 5t\right)\right\}(s) = -\left[\mathcal{L}\left\{e^{2t} \cos 5t\right\}(s)\right]' = -\left[\frac{s-2}{(s-2)^2 + 25}\right]'$$

$$= -\frac{[(s-2)^2 + 25] - (s-2) \cdot 2(s-2)}{[(s-2)^2 + 25]^2} = \frac{(s-2)^2 - 25}{[(s-2)^2 + 25]^2}.$$

Problem 22 solution:

We represent $t^n = t^n \cdot 1$ and apply (6) to get

$$\mathcal{L}\{t^n\}(s) = \mathcal{L}\{t^n \cdot 1\}(s) = (-1)^n \frac{d^n}{ds^n}\left[\mathcal{L}\{1\}(s)\right] = (-1)^n \frac{d^n\left(s^{-1}\right)}{ds^n}$$
$$= (-1)^n(-1)(-2)\cdots(-n)s^{-1-n} = \frac{(-1)^n(-1)^n 1\cdot 2\cdots n}{s^{n+1}} = \frac{n!}{s^{n+1}}\,.$$

Problem 25 solution:

Not written

Section 7.4 problems.

Problem 18 solution:

We have

$$\frac{3s^2+5s+3}{s^4+s^3} = \frac{3s^2+5s+3}{s^3(s+1)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s+1}\,. \qquad (7.2)$$

Multiplying this equation by $s+1$ and evaluating the result at $s=-1$ yields

$$\left.\frac{3s^2+5s+3}{s^3}\right|_{s=-1} = (s+1)\left(\frac{A}{s^3}+\frac{B}{s^2}+\frac{C}{s}\right) + D\bigg|_{s=-1} \qquad \Rightarrow \qquad D=-1\,.$$

We can find A by multiplying (7.2) by s^3 and substituting $s=0$.

$$\left.\frac{3s^2+5s+3}{s+1}\right|_{s=0} = A + Bs + Cs^2 + \left.\frac{Ds^3}{s+1}\right|_{s=0} \qquad \Rightarrow \qquad A=3\,.$$

Thus,

$$\frac{3s^2+5s+3}{s^3(s+1)} = \frac{3}{s^3} + \frac{B}{s^2} + \frac{C}{s} - \frac{1}{s+1}$$
$$\Rightarrow \qquad 3s^2+5s+3 = 3(s+1) + Bs(s+1) + Cs^2(s+1) - s^3\,. \qquad (7.3)$$

One can now compare the coefficients at s^3 and s to find B and C. Alternatively, differentiating (7.3) and evaluating the derivatives at $s = 0$ yields

$$6s + 5|_{s=0} = 5 = 3 + B(2s+1)|_{s=0} = 3 + B \qquad \Rightarrow \qquad B = 2\,.$$

(The last two terms in the right-hand side of (7.3) have zero derivative at $s = 0$.) Similarly, evaluating the second derivative in (7.3) at $s = 0$, we find that

$$6 = 2B + C(6s+2)|_{s=0} = 4 + 2C \qquad \Rightarrow \qquad C = 1\,.$$

Therefore,

$$\frac{3s^2 + 5s + 3}{s^4 + s^3} = \frac{3}{s^3} + \frac{2}{s^2} + \frac{1}{s} - \frac{1}{s+1}\,.$$

Problem 26 solution:

The partial fractions decomposition has the form

$$F(s) = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s-2}\,. \tag{7.5}$$

Multiplying (7.5) by s^3 and substituting $s = 0$ yields

$$\left.\frac{7s^3 - 2s^2 - 3s + 6}{s-2}\right|_{s=0} = -3 = A + Bs + Cs^2 + \left.\frac{Ds^3}{s-2}\right|_{s=0} = A\,.$$

Thus, $A = -3$. Multiplying (7.5) by $s - 2$ and evaluating the result at $s - 2$, we get

$$\left.\frac{7s^3 - 2s^2 - 3s + 6}{s^3}\right|_{s=2} = 6 = (s-2)\left[-\frac{3}{s^3} + \frac{B}{s^2} + \frac{C}{s}\right] + D\Big|_{s=2} = D\,.$$

So, $D = 6$ and (7.5) becomes

$$\frac{7s^3 - 2s^2 - 3s + 6}{s^3(s-2)} = -\frac{3}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{6}{s-2}\,.$$

Clearing the fractions yields

$$7s^3 - 2s^2 - 3s + 6 = -3(s-2) + Bs(s-2) + Cs^2(s-2) + 6s^3\,.$$

Matching the coefficients at s^3, we obtain $C + 6 = 7$ or $C = 1$. Finally, the coefficients at s^2 lead to $B - 2C = -2$ or $B = 0$. Therefore,

$$F(s) = \frac{7s^3 - 2s^2 - 3s + 6}{s^3(s-2)} = -\frac{3}{s^3} + \frac{1}{s} + \frac{6}{s-2}$$

and

$$\mathcal{L}^{-1}\{F(s)\}(t) = -\frac{3}{2}t^2 + 1 + 6e^{2t}.$$

Problem 30 solution:

Solving for $F(s)$ yields

$$F(s) = \frac{2s+5}{(s-1)(s^2+2s+1)} = \frac{A}{(s+1)^2} + \frac{B}{s+1} + \frac{C}{s-1}. \tag{7.6}$$

Thus, clearing fractions, we conclude that

$$2s + 5 = A(s-1) + B\left(s^2 - 1\right) + C(s+1)^2.$$

Substitution $s = 1$ into this equation yields $C = 7/4$. With $s = -1$, we get $A = -3/2$. Finally, substitution $s = 0$ results $5 = -A - B + C$ or $B = -A + C - 5 = -7/4$. Now we use the linearity of the inverse Laplace transform and obtain

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\}(t) &= -\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}(t) - \frac{7}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) + \frac{7}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) \\ &= -\frac{3}{2}te^{-t} - \frac{7}{4}e^{-t} + \frac{7}{4}e^{t}.\end{aligned}$$