Section 7.5 problems.

Problem 8 solution:

Applying the Laplace transform to both sides of the given equation and using Theorem 5 in Section 7.3 to express $\mathcal{L}\{y''\}$ and $\mathcal{L}\{y'\}$ in terms of Y, we obtain

$$\left(s^2Y-3\right)+4Y=\frac{8}{s^3}-\frac{4}{s^2}+\frac{10}{s}$$

$$\Rightarrow \qquad Y=\frac{1}{s^2+4}\left(3+\frac{8}{s^3}-\frac{4}{s^2}+\frac{10}{s}\right)$$

$$=\frac{3s^3+10s^2-4s+8}{s^3\left(s^2+2^2\right)}=\frac{2}{s^3}-\frac{1}{s^2}+\frac{2}{s}-2\frac{s}{s^2+2^2}+2\frac{2}{s^2+2^2}\,.$$

Taking now the inverse Laplace transform and using its linearity and Table 7.1 from Section 7.2 yields

$$y(t)=\mathcal{L}^{-1}\left\{\frac{2}{s^3}-\frac{1}{s^2}+\frac{2}{s}-2\frac{s}{s^2+2^2}+2\frac{2}{s^2+2^2}\right\}(t)=t^2-t+2-2\cos 2t+2\sin 2t\,.$$

Problem 22 solution:

Applying the Laplace transform to both sides of the given equation and using Theorem 5 in Section 7.3 to express $\mathcal{L}\{y''\}$ and $\mathcal{L}\{y'\}$ in terms of Y, we obtain

$$\left(s^2Y-2s+1\right)-6(sY-2)+5Y=\frac{1}{(s-1)^2}\,.$$

Solving for $Y(s)$ yields

$$Y(s)=\frac{1}{s^2-6s+5}\left[2s-13+\frac{1}{(s-1)^2}\right]$$

$$=\frac{2s^3-17s^2+28s-12}{(s-1)^2\left(s^2-6s+5\right)}=\frac{2s^3-17s^2+28s-12}{(s-1)^3(s-5)}\,.$$

Problem 26 solution:

Applying the Laplace transform to both sides of the given equation and using Theorem 5 in Section 7.3 to express $\mathcal{L}\{y'''\}$, $\mathcal{L}\{y''\}$ and $\mathcal{L}\{y'\}$ in terms of Y, we obtain

$$\left(s^3Y-s^2-4s+2\right)+4\left(s^2Y-s-4\right)+(sY-1)+Y=-\frac{12}{s}\,.$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{1}{s^3+4s^2+s-6}\left(s^2+8s+15-\frac{12}{s}\right)$$
$$= \frac{s^3+8s^2+15s-12}{s\left(s^3+4s^2+s-6\right)} = \frac{s^3+8s^2+15s-12}{s(s-1)(s+2)(s+3)}$$
$$= \frac{1}{s-1}+\frac{1}{s+3}-\frac{3}{s+2}+\frac{2}{s}.$$

Taking now the inverse Laplace transform leads to the solution

$$y(t) = e^t + e^{-3t} - 3e^{-2t} + 2\,.$$

Section 7.6 problems.

Problem 6 solution:

The function $g(t)$ equals zero until t reaches 2, at which point $g(t)$ jumps to $t+1$. We can express this jump by $(t+1)u(t-2)$. Hence,

$$g(t) = (t+1)u(t-2)$$

and, by formula (8),

$$\mathcal{L}\left\{g(t)\right\}(s) = e^{-2s}\mathcal{L}\left\{u\left[(t+1)+2\right]\right\}(s) = e^{-2s}\left(\frac{1}{s^2}+\frac{3}{s}\right) = \frac{e^{-2s}(3s+1)}{s^2}\,.$$

Problem 8 solution:

Observe from the graph that $g(t)$ is given by

$$\begin{cases} 0, & t < \pi/2, \\ \sin t, & t > \pi/2. \end{cases}$$

The function $g(t)$ equals zero until t reaches the point $\pi/2$, at which $g(t)$ jumps to the function $\sin t$. We can express this jump by $(\sin t)u(t-1)$. Hence

$$g(t) = (\sin t)u\left(t-\frac{\pi}{2}\right).$$

Taking the Laplace transform of both sides and using formula (8), we find that the Laplace transform of the function $g(t)$ is given by

$$
\begin{aligned}
\mathcal{L}\{g(t)\}(s) &= \mathcal{L}\left\{(\sin t)u\left(t-\frac{\pi}{2}\right)\right\}(s) \\
&= e^{-\pi s/2}\mathcal{L}\left\{\sin\left(t+\frac{\pi}{2}\right)\right\}(s) = e^{-\pi s/2}\mathcal{L}\{\cos t\}(s) = \frac{e^{-\pi s/2}s}{s^2+1}.
\end{aligned}
$$

Problem 9 solution:

Not given.

Problem 36 solution:

We take the Laplace transform of the both sides of the given equation and use the initial conditions, $y(0)=0$ and $y'(0)=1$ to obtain

$$
\begin{aligned}
\left[s^2Y(s)-1\right]+5sY(s)+6Y(s) &= \mathcal{L}\{tu(t-2)\}(s) \\
&= \mathcal{L}\{(t-2)u(t-2)\}(s)+2\mathcal{L}\{u(t-2)\}(s) \\
&= \frac{e^{-2s}}{s^2}+2\frac{e^{-2s}}{s} = \frac{e^{-2s}(2s+1)}{s^2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left(s^2+5s+6\right)Y(s) &= 1+\frac{e^{-2s}(2s+1)}{s^2} \\
\Rightarrow \qquad Y(s) &= \frac{1}{(s+2)(s+3)}+\frac{e^{-2s}(2s+1)}{s^2(s+2)(s+3)}.
\end{aligned}
$$

Using partial fractions decomposition yields

$$
\begin{aligned}
Y(s) &= \frac{1}{s+2}-\frac{1}{s+3}+e^{-2s}\left[\frac{1}{6s^2}+\frac{7}{36s}-\frac{3}{4(s+2)}+\frac{5}{9(s+3)}\right] \\
\Rightarrow \qquad y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s+2}-\frac{1}{s+3}+e^{-2s}\left[\frac{1}{6s^2}+\frac{7}{36s}-\frac{3}{4(s+2)}+\frac{5}{9(s+3)}\right]\right\}(t) \\
&= e^{-2t}-e^{-3t}+\left[\frac{7}{36}+\frac{t-2}{6}-\frac{3e^{-2(t-2)}}{4}+\frac{5e^{-3(t-2)}}{9}\right]u(t-2).
\end{aligned}
$$