X Find a formal solution to the given PDE:

$$\begin{cases} u_{xx} + u_{yy} = 0 & 0 < x < \pi, \; 0 < y < \pi \\ u(0,y) = u(\pi,y) = 0 & 0 < y < \pi \\ u(x,0) = \sin x + \sin 4x & 0 < x < \pi \\ u(x,\pi) = 0 & 0 < x < \pi \end{cases}$$

Let $u(x,y) = X(x)Y(y)$

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = k \quad \Rightarrow \begin{cases} X''(x) - kX(x) = 0 \\ Y''(y) + kY(y) = 0 \end{cases}$$

From $u(0,y) = u(\pi,y) = 0 \quad \Rightarrow X(0) = X(\pi) = 0$

$$\begin{cases} X''(x) - kX(x) = 0 \\ X(0) = X(\pi) = 0 \end{cases} \quad \Rightarrow X_n(x) = A_n \sin(nx)$$

$$k_n = -n^2, \quad n = 1, 2, \ldots$$

$$Y''(y) - n^2 Y(y) = 0$$

$$\lambda^2 - n^2 = 0 \quad \Rightarrow \lambda = \pm n$$

$$Y(y) = B_n e^{ny} + C_n e^{-ny} = D_n \cosh(ny) + E_n \sinh(ny)$$

$$\sinh(z) = \frac{e^z - e^{-z}}{2} \; ; \; \cosh(z) = \frac{e^z + e^{-z}}{2}$$

~~$Y_0(y) = A_0 + B_0 y$ (when $k=0$)~~

Next $u(x,\pi) = 0 \Rightarrow Y(\pi) = 0$

~~$\Rightarrow D_0 = -E_0$~~ and $A_n = -E_n \tanh(n\pi)$

or $Y_n(y) = F_n \sinh[n(y-\pi)]$

then

$$u_n(x,y) = X_n(x)Y_n(y) = A_n \sin(nx) F_n \sinh[n(y-\pi)]$$

$$u_n(x,y) = G_n \sin(nx) \sinh[n(y-\pi)]$$

$$\Rightarrow u(x,y) = \sum_{n=1}^{\infty} G_n \sin(nx) \sinh\left[n(y-\pi)\right]$$

The remaining nonhomogeneous BC's:

$$u(x,0) = \sum_{n=1}^{\infty} G_n \sin(nx) \sinh(-n\pi) = \sin x + \sin 4x$$

$$\Rightarrow \cancel{G_n = \frac{2}{\pi \sinh(-n\pi)} \int_0^{\pi}}$$

$$n=1 \Rightarrow G_1 \sin(x) \sinh(-\pi) = \sin x$$

$$G_4 \sin(4x) \sinh(-4\pi) = \sin 4x$$

$$\Rightarrow G_1 = \frac{1}{\sinh(-\pi)} \quad ; \quad G_4 = \frac{1}{\sinh(-4\pi)}$$

$$u(x,y) = \frac{\sin(x)}{\sinh(-\pi)} \sinh\left[(y-\pi)\right] + \frac{\sin(4x)}{\sinh(-4\pi)} \sinh\left[4(y-\pi)\right]$$

XI
$$\begin{cases} \dfrac{\partial u}{\partial t} = 7\dfrac{\partial^2 u}{\partial x^2} & 0<x<\pi,\ 0<t \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(\pi,t) = 0 & 0<t \\ u(x,0) = 1-\sin x & 0<x<\pi \end{cases}$$

Using the method of separation of variablele, we assume $u(x,t) = X(x)T(t)$

$$\Rightarrow \begin{cases} X''(x) - kX(x) = 0 \\ T'(t) - 7kT(t) = 0 \end{cases}$$

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial x}(0,t) = X'(0)T(t) = 0 \\ \dfrac{\partial u}{\partial x}(\pi,t) = X'(\pi)T(t) = 0 \end{array}\right\} \Rightarrow T(t)=0 \Rightarrow u(x,t)=0 \text{ trivial sol}$$

$$\Rightarrow X'(0) = X'(\pi) = 0$$

$$\Rightarrow \begin{cases} X''(x) - kX(x) = 0 \\ X'(0) = X'(\pi) = 0 \end{cases} \qquad \lambda = \pm\sqrt{k}$$

Case 1 $k>0$ $\Rightarrow X(x) = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}$; $X'(x) = \sqrt{k}(C_1 e^{\sqrt{k}x} - C_2 e^{-\sqrt{k}x})$

$X(0) = C_1 \cancel{+} C_2 = 0 \Rightarrow C_2 = +C_1$

$X(\pi) = (C_1 e^{\sqrt{k}\pi} - C_2 e^{-\sqrt{k}\pi})\sqrt{k} = C_1(e^{\sqrt{k}\pi} - e^{-\sqrt{k}\pi})\sqrt{k} = 0 \Rightarrow C_1 = 0$ $\Big\} \Rightarrow C_2 = 0$

$\Rightarrow X(x) = 0 \Rightarrow u(x,t) = 0$ (trivial sol)

Case 2 $k=0$

$X(x) = C_1 + C_2 x \Rightarrow X'(x) = C_2$

$$\left.\begin{array}{l} X(0) = C_2 \\ X(\pi) = C_2 = 0 \end{array}\right\} \Rightarrow C_2 = 0 \Rightarrow C_1 \in \mathbb{R} \quad \Rightarrow X(x) = C_0$$

Case 3 $k<0$ $\lambda = \pm i\sqrt{-k}$

$X(x) = C_1\cos(\sqrt{-k}x) + C_2\sin(\sqrt{-k}x)$

$X'(x) = -\sqrt{-k}C_1\sin(\sqrt{-k}x) + \sqrt{-k}C_2\cos(\sqrt{-k}x)$

$X'(0) = \sqrt{-k}C_2 = 0 \Rightarrow C_2 = 0$

$X'(\pi) = -\sqrt{-k}C_1\sin(\sqrt{-k}\pi) = 0 \Rightarrow \sqrt{-k}\pi = n\pi \quad n = 1, 2 \ldots$

$$\boxed{\Rightarrow X_n(x) = C_n\cos\frac{n\pi x}{\pi} = C_n\cos(nx) \qquad k = -n^2, \ n = 0, 1, 2 \ldots}$$

$$T'(t) + 7n^2\pi^2 T(t) = 0$$

$$T_n(t) = b_n e^{-7n^2 t}$$

$$\Rightarrow u_n(x,t) = X_n(x) T_n(t) = C_n \cos(nx)\, b_n e^{-7n^2 t}$$

$$\Rightarrow u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-7n^2 t} \cos(nx)$$

c: $u(x,0) = 1 - \sin x$

write Fourier cosine series

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx \qquad n = \cancel{0}\,1, \dots \qquad a_0 = \dots = 2 - \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(1-\sin x)\cos(nx)\,dx = \frac{2}{\pi}\left[\int_0^{\pi}\cos(nx)\,dx - \int_0^{\pi}\sin(x)\cos(nx)\,dx\right]$$

$$\int_0^{\pi}\cos(nx)\,dx = \frac{1}{n}\sin(nx)\Big|_0^{\pi} = \frac{1}{n}\left[\sin(n\pi) - \sin(0)\right] = 0$$

$$\int_0^{\pi}\sin(x)\cos(nx)\,dx \quad \Bigg\{ \text{ see tables: } \int \sin au \cos bu\, du = -\frac{\cos(a+b)u}{2(a+b)} - \frac{\cos(a-b)u}{2(a-b)}$$

$$\int_0^{\pi}\sin(x)\cos(nx)\,dx = \left[-\frac{\cos[(n+1)x]}{2(n+1)} - \frac{\cos[(1-n)x]}{2(1-n)}\right]_0^{\pi} =$$

$a = 1$
$b = n$

$$= -\frac{\cos[(n+1)\pi]}{2(n+1)} - \frac{\cos[(1-n)\pi]}{2(1-n)} + \frac{1}{2(n+1)} + \frac{1}{2(1-n)}$$

$$a_n = \frac{2}{\pi}\left[\frac{\cos[(n+1)\pi]}{2(n+1)} + \frac{\cos[(1-n)\pi]}{2(1-n)} + \frac{1}{2(n+1)} + \frac{1}{2(1-n)}\right]$$

$k \geq 1$

$$n = 2k+1 \qquad a_{2k+1} = \frac{2}{\pi}\left[\frac{\cancel{2}(1-\cancel{n}+\cancel{n}+1)}{\cancel{2}(1-n^2)}\right] = \frac{4}{\pi(1-n^2)} = \frac{4}{\pi[1-(2k+1)^2]}$$

$$n = 2k \qquad a_{2k} = \frac{2}{\pi}\left[\frac{(-1)^{2k+1}}{2(n+1)} + \frac{(-1)^{2k+1}}{2(1-n)} + \frac{2}{1-n^2}\right] =$$

$$= \frac{2}{\pi}\left[\cdot\frac{2}{1-n^2}\left[(-1)^{2k+1}+1\right]\right] =$$

$$a_{2k} = \frac{4}{\pi(1-n^2)}\left[(-1)^{2k+1}+1\right] = 0$$

a_1 calculate separatelly

$$u(x,t) = \cancel{\sum_{k=1}^{\infty} \frac{4}{\pi(1-4k^2)}\left[(-1)^{2k+1}\right] e^{-7n^2 t}\cos(nx)} =$$

$$= 1 - \frac{2}{\pi} + 2e^{-7t}\cos x + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{4k^2-1} e^{-28k^2 t}\cos(2kx)$$

XII find at least the first ~~nonzero~~ four nonzero terms about $x=0$

$$xy'' + y' - 4y = 0 \quad , \; x>0$$

$$y'' + \frac{1}{x}y' - \frac{4}{x}y = 0$$

Not analytic at $x=0$

$$p(x) = \frac{1}{x} \; ; \; q(x) = -\frac{4}{x}$$

$$\left.\begin{array}{l} xp(x) = 1 \\ x^2 q(x) = -4x \end{array}\right\} \Rightarrow \text{analytic everywhere}$$

$\Rightarrow$ apply the method of Frobenius

The indicial equation: $[r(r-1) + p_0 r + q_0] = 0$

$$\left.\begin{array}{l} p_0 = \lim\limits_{x\to 0} xp(x) = 1 \\ q_0 = \lim\limits_{x\to 0}(-4x) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} r(r-1) + r = 0 \\ r^2 - r + r = 0 \Rightarrow r_{1,2} = 0 \end{array}$$

Consider $w(r_1, x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} = y(x)$

$$y'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

By substitution

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - 4\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=1}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=1}^{\infty}(n+r)a_n x^{n+r-1} - 4\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$k = n-1$, $n = k+1$; $k = n-1$; $k = n$

$$\sum_{k=0}^{\infty}(k+r+1)(k+r)a_{k+1} x^{k+r} + \sum_{k=0}^{\infty}(k+r+1)a_{k+1} x^{k+r} - 4\sum_{k=0}^{\infty} a_k x^{k+r} = 0$$

$$\sum_{k=0}^{\infty}\left[(k+r+1)(k+r)a_{k+1}+(k+r+1)a_{k+1}-4a_k\right]x^{k+r}=0$$

$r=0$

$$\sum_{k=0}^{\infty}\left[k(k+1)a_{k+1}+(k+1)a_{k+1}-4a_k\right]x^{k}=0$$

$$\sum_{k=0}^{\infty}\left[(k^2+2k+1)a_{k+1}-4a_k\right]x^{k}=0$$

$$\sum_{k=0}^{\infty}\left[(k+1)^2a_{k+1}-4a_k\right]x^{k}=0 \Rightarrow a_{k+1}=\frac{4a_k}{(k+1)^2}$$

$k=0 \Rightarrow 1^2a_1-4a_0=0 \Rightarrow a_1=4a_0$

$k=1 \Rightarrow a_2=\frac{4a_1}{2^2}=a_1=4a_0$

$k=2 \Rightarrow a_3=\frac{4a_2}{9}=\frac{4}{9}\cdot 4a_0=\frac{16a_0}{9}$

$k=3 \Rightarrow a_4=\frac{4a_3}{16}=\frac{\cancel{4}}{\cancel{16}}\cdot\frac{\cancel{16}a_0}{9}=\frac{4}{9}a_0$

$\Rightarrow w(0,x)=a_0\left\{1+4x+4x^2+\frac{16}{9}x^3+\frac{4}{9}x^4+\dots\right\}$

XIII Determine an inverse Laplace transformation of:

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}(3s^2-s+2)}{(s-1)(s^2+1)}\right\}=f(t-a)u(t-a)=$$

$$\mathcal{L}^{-1}\left\{\frac{3s^2-s+2}{(s-1)(s^2+1)}\right\}=f(t)$$

$$\frac{3s^2-s+2}{(s-1)(s^2+1)}=\frac{A}{s-1}+\frac{Bs+C}{s^2+1}$$

$3s^2-s+2=A(s^2+1)+(Bs+C)(s-1)$

$3s^2-s+2=As^2+A+Bs^2+Cs-Bs-C$

$3s^2-s+2=(A+B)s^2+s(C-B)+A-C \Rightarrow \begin{cases}A+B=3\\C-B=-1\\A-C=2\end{cases}$ $\quad\begin{array}{l}A+C=2\\A-C=2\\\hline 2A=4\end{array}$

$A=2$
$C=0$
$B=1$

$$\Rightarrow \frac{3s^2-s+2}{(s-1)(s^2+1)}=\frac{2}{s-1}+\frac{s}{s^2+1}\ ;\quad \mathcal{L}^{-1}\left\{\frac{2}{s-1}+\frac{s}{s^2+1}\right\}=2e^{t}+\cos t=f(t)$$

$$\boxed{f(t-a)u(t-a)=\left[2e^{t-1}+\cos(t-1)\right]u(t-1)}$$