I.) $y'' - y' + y = 0$

a.) $y = e^{\lambda x}$, $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$

$\lambda^2 - \lambda + 1 = 0$

$\lambda = \frac{1 \pm \sqrt{1-4}}{2}$

$= \frac{1}{2} \pm \frac{\sqrt{3}}{2} i \left(\begin{matrix} \alpha = 1/2 \\ \beta = \sqrt{3}/2 \end{matrix} \right)$

$y_h = \boxed{c_1 e^{1/2 x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{1/2 x} \sin\left(\frac{\sqrt{3}}{2}x\right)}$

b.) $x^2 y'' - 5xy' + 25y = 0$

$y = x^m$, $y' = mx^{m-1}$, $y'' = (m^2-m)x^{m-2}$

$(m^2-m)x^m - 5mx^m + 25x^m = 0$

$m^2 - 6m + 25 = 0$

$m = \frac{6 \pm \sqrt{36-100}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i$

$\boxed{y_h = c_1 x^3 \cos(4 \ln x) + c_2 x^3 \sin(4 \ln x)}$

II.) 1ST SOLVE $y'' - 4y' + 3y = 0$

a.) (SAME ASSUMPTIONS AS I a)

$\lambda^2 - 4\lambda + 3 = 0$

$(\lambda - 1)(\lambda - 3) = 0$

$\lambda = 1, 3$

$\boxed{y_h = c_1 e^x + c_2 e^{3x}}$

CONSIDER $y'' - 4y' + 3y = 2e^{3x}$

$y_{p_1} = Axe^{3x}$

$y_{p_1}' = Ae^{3x} + 3Axe^{3x}$

$y_{p_1}'' = 3Ae^{3x} + 3Ae^{3x} + 9Axe^{3x}$

$= 6Ae^{3x} + 9Axe^{3x}$

$6Ae^{3x} + 9Axe^{3x} - 4Ae^{3x} - 12Axe^{3x} + 3Axe^{3x} = 0$

$2Ae^{3x} = 2e^{3x}$

$2A = 2 \Rightarrow \boxed{A = 1}$

$\boxed{y_{p_1} = xe^{3x}}$

CONSIDER $y'' - 4y' + 3y = \sin x$

$y_{p_2} = A\sin x + B\cos x$

$y_{p_2}' = A\cos x - B\sin x$

$y_{p_2}'' = -A\sin x - B\cos x$

$-A\sin x - B\cos x - 4A\cos x + 4B\sin x + 3A\sin x + 3B\cos x = \sin x$

$-A + 4B + 3A = 1 \Rightarrow 2A + 4B = 1$

$-B - 4A + 3B = 0 \Rightarrow -4A + 2B = 0$

$\Rightarrow 4A + 8B = 2$

$-4A + 2B = 0$

$10B = 2 \Rightarrow \boxed{B = 1/5}$

$2A + 4/5 = 1 \Rightarrow 2A = 1/5 \Rightarrow \boxed{A = 1/10}$

$\boxed{y_{p_2} = 1/10 \sin x + 1/5 \cos x}$

$\boxed{y_p = c_1 e^x + c_2 e^{3x} + xe^{3x} + 1/10 \sin x + 1/5 \cos x}$

b.) $y_h = c_1 e^x + c_2 e^x \ln x$

$y_p = v_1 e^x + v_2 e^x \ln x$

$W[y_1, y_2] = e^x [e^x \ln x + 1/x\, e^x] - e^x \ln x\, e^x$

$= e^{2x} \ln x + 1/x\, e^{2x} - e^{2x} \ln x = 1/x\, e^{2x}$

$a(x) = x$

$f(x) = xe^x$

$v_1 = -\int \frac{y_2 f(x)}{a(x) W[y_1, y_2]} dx = -\int \frac{e^x \ln x\,(xe^x)}{x(1/x\, e^{2x})} dx$

$= -\int x \ln x \, dx$

$u = \ln x \quad dv = x\,dx$

$du = 1/x\,dx \quad v = x^2/2$

$v_1 = -\left[\frac{x^2}{2} \ln x - \int \frac{x}{2} dx\right] = \boxed{-\frac{x^2}{2} \ln x + x^2/4}$

$v_2 = \int \frac{y_1 f(x)}{a(x) W[y_1, y_2]} dx = \int \frac{e^x (xe^x)}{x(1/x\, e^{2x})} dx$

$= \int x\,dx = \boxed{x^2/2}$

$y_p = \left[-\frac{x^2}{2} \ln x + x^2/4\right] e^x + x^2/2 (e^x \ln x)$

$= \boxed{x^2/4\, e^x}$

$\boxed{y_g = c_1 e^x + c_2 e^x \ln x + x^2/4\, e^x}$

III.) a.) BERNOULLI

$\frac{dy}{dx} + 2y = y^2$

$y^{-2} \frac{dy}{dx} + 2y^{-1} = 1$

$z = y^{-1} \quad \frac{dz}{dx} = -y^{-2} \frac{dy}{dx}$

$-\frac{dz}{dx} = y^{-2} \frac{dy}{dx}$

$-\frac{dz}{dx} + 2z = 1$

$\frac{dz}{dx} - 2z = -1$ (LINEAR)

$\mu(x) = e^{\int -2dx} = e^{-2x}$

$\underbrace{e^{-2x}\frac{dz}{dx} - 2e^{-2x}z}_{} = -e^{-2x}$

$\frac{d}{dx}(ze^{-2x}) = -e^{-2x}$

$ze^{-2x} = \frac{1}{2}e^{-2x} + C$

$z = 1/2 + Ce^{2x}$

$\boxed{y^{-1} = 1/2 + Ce^{2x}}$

b.) $\underset{M}{2xy}\,dx + \underset{N}{(3x^2+3)}\,dy = 0$

$\frac{\partial M}{\partial y} = 2x \neq \frac{\partial N}{\partial x} = 6x$ (NOT EXACT)

$Q = \dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{6x-2x}{2xy} = \frac{4x}{2xy} = 2/y$ (DEPENDS ONLY ON y)

$\mu(y) = e^{\int 2/y\,dy} = e^{2\ln y} = e^{\ln y^2} = y^2$

$2xy^3\,dx + (3x^2y^2 + 3y^2)\,dy = 0$

$\partial M/\partial y = 6xy^2 = \partial N/\partial x = 6xy^2 \ni$ EXACT

$\frac{\partial f}{\partial x} = 2xy^3$

$f(x,y) = x^2y^3 + g(y)$

$\frac{\partial f}{\partial y} = 3x^2y^2 + g'(y) = N = 3x^2y^2 + 3y^2$

$g'(y) = 3y^2 \ni g(y) = y^3$

$f(x,y) = x^2y^3 + y^3$

SOLUTION $\boxed{x^2y^3 + y^3 = C}$

IV.) a.) $\mathcal{L}\{f\} = \int_0^5 2e^{-st}\,dt + \int_5^\infty 0e^{-st}\,dt$

$= -\frac{2}{s}e^{-st}\Big]_0^5 = \boxed{-\frac{2}{s}e^{-5s} + \frac{2}{s}}$

b.) $\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$

$\mathcal{L}\{e^{2t}\sin 3t\} = \frac{3}{(s-2)^2+9}$

$\ni \mathcal{L}\{te^{2t}\sin 3t\} = -\frac{d}{ds}\left[\frac{3}{(s-2)^2+9}\right]$

$= \boxed{-\left[\frac{-3}{[(s-2)^2+9]^2}\right][2(s-2)]}$

V.)

a) $\hat{f} = \frac{7s^2 - 11s - 5}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$

$7s^2 - 11s - 5 = A(s-1)(s+2) + B(s+2) + C(s-1)^2$

$s=1$ $\quad -9 = 3B \ni \boxed{B = -3}$

$s=-2$ $\quad 45 = 9C \ni \boxed{C = 5}$

$s=0$ $\quad -5 = -2A + 2B + C$

$-5 = -2A - 6 + 5$

$-4 = -2A \ni \boxed{A = 2}$

$\hat{f} = \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{5}{s+2}$

$f(t) = \boxed{2e^t - 3te^t + 5e^{-2t}}$

b.) $\mathcal{L}\{y\} = \hat{y}$, $\mathcal{L}\{y'\} = s\hat{y} - 2$

$\mathcal{L}\{y''\} = s^2\hat{y} - 2s - 1$

$s^2\hat{y} - 2s - 1 + 4s\hat{y} - 8 + 20\hat{y} = 0$

$(s^2 + 4s + 20)\hat{y} = 2s + 9$

$\hat{y} = \frac{2s+9}{s^2+4s+20} = \frac{2s+9}{s^2+4s+4+16}$

$= \frac{2s+9}{(s+2)^2+16}$

$= \frac{2s+4+5}{(s+2)^2+16}$

$= \frac{2(s+2)}{(s+2)^2+16} + \frac{5}{(s+2)^2+16}$

$= \frac{2(s+2)}{(s+2)^2+16} + \frac{5}{4}\left[\frac{4}{(s+2)^2+16}\right]$

SOLUTION: $f(t) = \boxed{2e^{-2t}\cos 4t + \frac{5}{4}e^{-2t}\sin 4t}$

I.) a.) $y'' + \frac{(x-1)}{x^2+4}y' + \frac{3}{x^2+4}y = 0$

$$y'' = \frac{(1-x)}{x^2+4}y' - \frac{3}{x^2+4}y$$

$y(2) = 1, \quad y'(2) = 3$

$$y''(2) = \frac{-1}{8}y'(2) - \frac{3}{8}y(2)$$

$$= -\tfrac{1}{8}(3) - \tfrac{3}{8}(1) = \boxed{-3/4}$$

$$y \sim \sum_{n=0}^{\infty} \frac{y^{(n)}(2)(x-2)^n}{n!} = \boxed{1 + 3(x-2) - \frac{3}{4}\frac{(x-2)^2}{2} + \cdots}$$

b.) $y = \sum_{n=0}^{\infty} a_n x^n$

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

$n-2=k \ni n=k+2$ $\quad n=k$ $\quad n=k$

$$\sum_{k=\cancel{-2}\,0}^{\infty} (k+2)(k+1)a_{k+2}x^k + \sum_{k=0}^{\infty} k(k-1)a_k x^k + \sum_{k=0}^{\infty} 3a_k x^k = 0$$

$$(k+2)(k+1)a_{k+2} = -k(k-1)a_k - 3a_k$$

$$a_{k+2} = \frac{-k^2+k-3}{(k+2)(k+1)}a_k$$

$k=0$ $\quad a_2 = -\frac{3}{2}a_0$

$k=1$ $\quad a_3 = \frac{-3}{6}a_1 = \frac{-a_1}{2}$

$k=2$ $\quad a_4 = \frac{-5}{12}a_2 = \frac{15}{24}a_0$

$k=3$ $\quad a_5 = \frac{-9}{20}a_3 = \frac{9a_1}{40}$

$$y = a_0 + a_1x - \tfrac{3}{2}a_0x^2 - \tfrac{a_1}{2}x^3 + \tfrac{15}{24}a_0x^4 + \tfrac{9}{40}a_1x^5 + \cdots$$

$$= \boxed{a_0\left[1 - \tfrac{3}{2}x^2 + \tfrac{15}{24}x^4 + \cdots\right] + a_1\left[x - \tfrac{1}{2}x^3 + \tfrac{9}{40}x^5 + \cdots\right]}$$

VII.) a.) $L=4$ $\quad f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left[\frac{n\pi x}{4}\right]$

$$a_0 = \frac{2}{4}\left[\int_0^2 0\,dx + \int_2^4 2\,dx\right] = \frac{1}{2}\Big[2x\Big]_2^4 = \frac{1}{2}[8-4] = \boxed{2}$$

$$a_n = \frac{2}{4}\left[\int_0^2 (0)\cos\left[\frac{n\pi x}{4}\right]dx + \int_2^4 2\cos\left[\frac{n\pi x}{4}\right]dx\right]$$

$$= \frac{1}{2}\left[\frac{8}{n\pi}\sin\left[\frac{n\pi x}{4}\right]\right]_2^4 = \boxed{\frac{4}{n\pi}\sin\left[\frac{n\pi}{2}\right]}$$

$n=1$ $\quad a_1 = \frac{4}{\pi}$ $\qquad$ $n=5$ $\quad a_5 = \frac{4}{5\pi}$

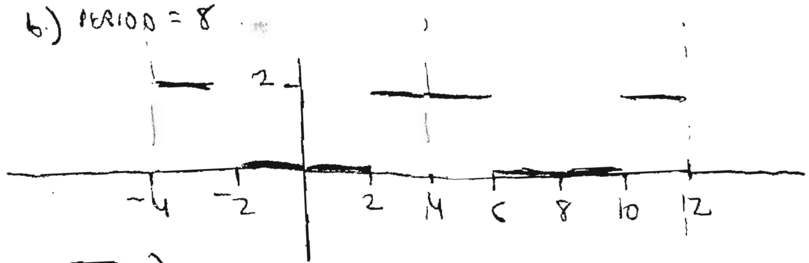
$n=2$ $\quad a_2 = 0$

$n=3$ $\quad a_3 = -\frac{4}{3\pi}$

$n=4$ $\quad a_4 = 0$

$$f(x) \sim \boxed{\frac{1}{2} + \frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right) - \frac{4}{3\pi}\cos\left(\frac{3\pi x}{4}\right) + \frac{4}{5\pi}\cos\left(\frac{5\pi x}{4}\right) + \cdots}$$

b.) PERIOD = 8



VIII.) $y'' - 4y' + \lambda y = 0$

$y = e^{rx}$

$r^2 - 4r + \lambda = 0$

$$r = \frac{4 \pm \sqrt{16-4\lambda}}{2} = \frac{4 \pm 2\sqrt{4-\lambda}}{2}$$

$$= 2 \pm \sqrt{4-\lambda}$$

CASE 1 $\quad 4-\lambda = 0 \ni \lambda = 4$

$r=2$ DOUBLE ROOT

$y = c_1e^{2x} + c_2xe^{2x}$

$0 = y(0) = c_1$

$0 = y(1) = c_2e^2 \ni c_2 = 0 \ni y \equiv 0$ TRIVIAL

CASE 2 $\quad 4-\lambda > 0 \quad 4-\lambda = k^2 \ni r = 2 \pm k$

$y = c_1e^{(2+k)x} + c_2e^{(2-k)x}$

$0 = y(0) = c_1 + c_2 \ni c_2 = -c_1$

$0 = y(1) = c_1e^{2+k} - c_1e^{2-k}$

$0 = c_1[e^{2+k} - e^{2-k}] \ni c_1 = 0, c_2 = 0$ (TRIVIAL)

CASE 3 $\quad 4-\lambda < 0 \quad 4-\lambda = -k^2 \ni r = 2 \pm ki$

$y = c_1e^{2x}\cos kx + c_2e^{2x}\sin kx$

$0 = y(0) = c_1$

$0 = y(1) = c_2e^2\sin k$

$\sin k = 0 \ni k = n\pi \ni \boxed{\lambda_n = 4 + n^2\pi^2}$

$(n = 1, 2, \cdots)$ $\qquad$ (EIGENVALUES)

$\boxed{y_n = c_ne^{2x}\sin(n\pi x)}$ (EIGENFUNCTIONS)

IX.) $u(x,t) = XT$

$XT'' = 4X''T$

$$\frac{T''}{4T} = \frac{X''}{X} = \lambda$$

$\boxed{X'' - \lambda X = 0}$, $\boxed{X'(0) = 0}$, $\boxed{X'(3) = 0}$

$T'' - 4\lambda T = 0$

$X = e^{rx}$

$r^2 - \lambda = 0 \ni r = \pm\sqrt{\lambda}$

$\underline{\lambda = 0}$ $r = 0$ DOUBLE ROOT

$X = c_1 + c_2 x$

$X' = c_2$

$0 = X'(0) = c_2$

$0 = X'(3) = c_2$

$\underline{\lambda = 0}$ $\underline{\text{IS}}$ AN EIGENVALUE W/EIGENFUNCTION

$X_0 = c_0$

$\underline{\lambda < 0}$ $\lambda = -k^2 \ni r = \pm ki$

$X = c_1 \cos kx + c_2 \sin kx$

$X' = -kc_1 \sin kx + kc_2 \cos kx$

$0 = X'(0) = kc_2 \ni c_2 = 0$

$0 = X'(3) = -kc_1 \sin 3k$

$\sin 3k = 0 \ni 3k = n\pi$

$\ni k = n\pi/3$

$\boxed{X_n = c_n \cos\left[\frac{n\pi x}{3}\right]}$ $\lambda_n = \frac{-n^2\pi^2}{9}$ $(n = 0, 1, \ldots)$

$T'' + \frac{4n^2\pi^2}{9} T = 0$

$T = e^{rt}$

$r^2 + \frac{4n^2\pi^2}{9} = 0$

$r = \frac{2}{3} n\pi i$

$\boxed{T_n = c_1 \cos\left[\frac{2}{3} n\pi t\right] + c_2 \sin\left[\frac{2}{3} n\pi t\right]}$

$u(x,t) = \sum_{n=0}^{\infty} \left[A_n \cos\left[\frac{n\pi x}{3}\right] \cos\left[\frac{2n\pi t}{3}\right] + B_n \cos\left[\frac{n\pi x}{3}\right] \sin\left[\frac{2n\pi t}{3}\right]\right]$

$u(x,0) = \sum_{n=0}^{\infty} A_n \cos\left[\frac{n\pi x}{3}\right] = 5\cos\left[\frac{7\pi x}{3}\right] - 11\cos\left[\frac{13\pi x}{3}\right]$

$\underline{n=7}$ $\boxed{A_7 = 5}$ $\underline{n=13}$ $\boxed{A_{13} = -11}$ THE OTHER A_n's $= 0$

$u_t(x,t) = \sum_{n=0}^{\infty} \left[-\frac{2n\pi}{3} A_n \cos\left[\frac{n\pi x}{3}\right] \sin\left[\frac{2n\pi t}{3}\right] + \frac{2n\pi}{3} B_n \cos\left[\frac{n\pi x}{3}\right] \cos\left[\frac{2n\pi t}{3}\right]\right]$

$u_t(x,0) = \sum_{n=0}^{\infty} \frac{2n\pi}{3} B_n \cos\left[\frac{n\pi x}{3}\right] = 2\cos\left[\frac{5\pi x}{3}\right] - 3\cos\left[\frac{11\pi x}{3}\right]$

$\underline{n=5}$ $\frac{10\pi}{3} B_5 = 2 \ni \boxed{B_5 = \frac{6}{10\pi}}$

$\underline{n=11}$ $\frac{22\pi}{3} B_{11} = -3 \ni \boxed{B_{11} = \frac{-9}{22\pi}}$ THE OTHER B_n's $= 0$

$$\boxed{u(x,t) = 5\cos\left[\frac{7\pi x}{3}\right]\cos\left[\frac{14\pi t}{3}\right] - 11\cos\left[\frac{13\pi x}{3}\right]\cos\left[\frac{26\pi t}{3}\right] + \frac{6}{10\pi}\cos\left[\frac{5\pi x}{3}\right]\sin\left[\frac{10\pi t}{3}\right] - \frac{9}{22\pi}\cos\left[\frac{11\pi x}{3}\right]\sin\left[\frac{22\pi t}{3}\right]}$$