MIDTERM 1 SUMMER 2009 - MA 221

Problem (1): (a) separable equation

(b) eq of type $G(ax+by)$

to solve: Make substitution $z = x+y$

(c) homogeneous first order

to solve: substitution $z = y/x$

(d) Bernoulli equation

to solve: substitution $z = y^3$

(e) second order constant coefficients - nonhomogeneous

to solve: use superposition principle: $f_1(t) = e^{3t}$, $f_2(t) = \tan t + 1$

- use undetermined coefficients with $f_1(t)$
- use variable coefficient method with $f_2(t)$

(f) Euler-Cauchy equation

to solve: use $t = e^s \to$ obtain constant coefficient eq.

or search for sol of type $y = t^r$

Problem (2)

Bernoulli equation. We make substitution $\underline{z = y^{-2}}$

$$\Rightarrow \frac{dz}{d\theta} = -2y^{-3}\frac{dy}{d\theta} \Rightarrow -2y^{-3}\frac{dy}{d\theta} = \frac{dz}{d\theta}$$

multiply original equation with $(-2)y^{-3}$ to obtain:

$$\underbrace{-2y^{-3}\frac{dy}{d\theta}}_{=\frac{dz}{d\theta}} - 2\theta - 2\underbrace{y^{-2}}_{=z} = 0$$

$$\Rightarrow \frac{dz}{d\theta} - 2z = 2\theta$$

Integrating factor $\mu(\theta) = e^{\int -2d\theta} = e^{-2\theta}$

$$\Rightarrow e^{-2\theta}\frac{dz}{d\theta} - 2e^{-2\theta}z = 2\theta e^{-2\theta}$$

$$\Rightarrow \frac{d}{d\theta}\left(e^{-2\theta}z\right) = 2\theta e^{-2\theta}$$

$$\Rightarrow e^{-2\theta}z = \int 2\theta e^{-2\theta} = \int -\theta\left(e^{-2\theta}\right)' d\theta =$$

$$= -\theta e^{-2\theta} + \int e^{-2\theta} d\theta =$$

$$= -\theta e^{-2\theta} - \frac{1}{2}e^{-2\theta} + C$$

$$\Rightarrow z = -\theta - \frac{1}{2} + Ce^{2\theta}$$

substitute back: $z = y^{-2}$

implicit solution

$$\Rightarrow y^{-2} = Ce^{2\theta} - \theta - \frac{1}{2} \Rightarrow y^2(\theta)\left(Ce^{2\theta} - \theta - \frac{1}{2}\right) = 1$$

or

explicit sol: $y_1(\theta) = \dfrac{1}{\sqrt{Ce^{2\theta} - \theta - \frac{1}{2}}}$

$y_2(\theta) = -$ same thing

Problem (3)

Solution A: The equation is exact; Rewrite:

$$\underbrace{(x-y-1)}_{M}dx - \underbrace{(x+y+5)}_{N}dy = 0$$

$$\frac{\partial M}{\partial y} = -1 \quad ; \quad \frac{\partial N}{\partial x} = -1 \Rightarrow \text{eq is exact.}$$

Let F such that $\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$ or

$$\frac{\partial F}{\partial x} = x-y-1 \quad ; \quad \frac{\partial F}{\partial y} = -x-y-5$$

$\Downarrow$

$F(x,y) = \frac{x^2}{2} - xy - x + c(y)$. From $\Rightarrow$

$$\cancel{-x} + c'(y) = \cancel{-x} - y - 5 \Rightarrow c'(y) = -y-5$$

$$c(y) = -\frac{y^2}{2} - 5y$$

So the solution is $F(x,y) = C$ or

$$\boxed{\frac{x^2}{2} - xy - x - \frac{y^2}{2} - 5y = C}$$

we may rewrite as:

$$x^2 - y^2 - 2xy - 2x - 10y = C$$

(For solution B turn page)

Solution B: We may also solve this problem using a translation.
Let's make substitution $x = u+h$, $y = v+k$ where h, k are sol

$$\begin{cases} h-k-1=0 \\ h+k+5=0 \end{cases}$$

$2h + 4 = 0 \Rightarrow \underline{h=-2}$; $\underline{k = h-1 = -3}$

So transformation is: $\boxed{\begin{array}{l} x = u-2 \\ y = v-3 \end{array}}$

eq. Becomes:

$$\frac{dv}{du} = \frac{u-2-v+3-1}{u-2+v-3+5} = \frac{u-v}{u+v} = \frac{1-\frac{v}{u}}{1+\frac{v}{u}}$$ (which is homogeneous)

So transformation: $z = \frac{v}{u}$ or $v(u) = z(u) \cdot u \Rightarrow \underline{\frac{dv}{du} = \frac{dz}{du} \cdot u + z}$

$\Rightarrow$ eq is now:

$$\frac{dz}{du} u + z = \frac{1-z}{1+z}$$ which is separable:

$$\frac{dz}{du} \cdot u = \frac{1-z-z-z^2}{1+z} = \frac{1-2z-z^2}{1+z}$$

$$\Rightarrow \frac{1+z}{1-2z-z^2} dz = \frac{1}{u} du$$. Integrate:

$$-\frac{1}{2}\int \frac{-2(1+z)}{1-2z-z^2} dz = \log u + C$$

$$\Rightarrow \ln\left|1-2z-z^2\right| = -2\ln u + c = \ln\frac{1}{u^2} + c$$

$$\Rightarrow 1-2z-z^2 = C \cdot \frac{1}{u^2}$$ substitute back $z = \frac{v}{u} = \frac{y+3}{x+2}$

$$\Rightarrow \boxed{1 - 2\frac{(y+3)}{x+2} - \left(\frac{y+3}{x+2}\right)^2 = \frac{C}{(x+2)^2}}$$ which can be shown is identical with the other sol

Problem (4) double root $r_1 = r_2 = -1$

homogeneous solutions : $y_1 = e^{-\theta}$

$y_2 = \theta e^{-\theta}$

We look for particular sol of the type :

$y_p(\theta) = A\cos\theta + B\sin\theta$

$y_p'(\theta) = -A\sin\theta + B\cos\theta$

$y_p''(\theta) = -A\cos\theta - B\sin\theta$

substitute into eq :

$\cancel{-A\cos\theta} \cancel{-B\sin\theta} \underline{-2A\sin\theta} + \underline{2B\cos\theta} \cancel{+A\cos\theta} \cancel{+B\sin\theta} = \underline{2\cos\theta}$

$\Rightarrow -2A = 0 \Rightarrow A = 0$
$2B = 2 \Rightarrow B = 1$
$\Rightarrow y_p(\theta) = \sin\theta$

solution is : $y(\theta) = C_1 e^{-\theta} + C_2 \theta e^{-\theta} + \sin\theta$

Now to find C_1, C_2 :

$y(0) = C_1 = 3 \Rightarrow \boxed{C_1 = 3}$

$y'(\theta) = -C_1 e^{-\theta} + C_2 e^{-\theta} - C_2 \theta e^{-\theta} + \cos\theta$

$y'(0) = -C_1 + C_2 + 1 = 0 \Rightarrow C_2 = C_1 - 1 = 3 - 1 = 2$

So :

$$\boxed{y(\theta) = 3e^{-\theta} + 2\theta e^{-\theta} + \sin\theta}$$

Problem (5) complex solutions $r_{1,2} = \pm i$

so homogeneous solutions $y_1(x) = \cos x$

$y_2(x) = \sin x$

Look for $v_1(x), v_2(x)$ in the $y_p(x) = v_1(x) \cdot \cos x + v_2(x) \sin x$

Solutions to:

$$\begin{cases} v_1' \cos x + v_2' \sin x = 0 & \Big| \cdot \sin x \\ v_1'(-\sin x) + v_2' \cos x = \dfrac{1}{\cos^3 x} & \Big| \cdot \cos x \end{cases}$$

$$\Rightarrow \begin{cases} v_1' \sin x \cos x + v_2' \sin^2 x = 0 \\ -v_1' \sin x \cos x + v_2' \cos^2 x = \dfrac{1}{\cos^2 x} \end{cases}$$

$$\boxed{v_2' = \frac{1}{\cos^2 x}}$$

$$\Rightarrow v_2(x) = \int \frac{1}{\cos^2 x} \, dx = \underline{\underline{\tan x}}$$

$$v_1'(x) = -v_2' \operatorname{tg} x = -\frac{1}{\cos^2 x} \cdot \frac{\sin x}{\cos x} = -\frac{\sin x}{\cos^3 x}$$

$$\Rightarrow v_1(x) = \int -\frac{\sin x}{\cos^3 x} \, dx = \int \frac{(\cos x)' \, dx}{\cos^3 x} \xlongequal{t = \cos x} \int \frac{1}{t^3} \, dt =$$

$$= -\frac{1}{2} \frac{1}{t^2} \xlongequal{\cos x = t} -\frac{1}{2\cos^2 x}$$

So:

$$y(x) = c_1 \cos x + c_2 \sin x + \operatorname{tg} x \sin x - \frac{1}{2\cos x}$$

Problem ⑥

$$y''' + 3y'' + 5y' + 3y = 0$$

$$r^3 + 3r^2 + 5r + 3 = 0$$

Verify $r = -1$ is a solution $\Rightarrow$ $\boxed{y_1(t) = e^{-t}}$

$$r^3 + r^2 + 2r^2 + 2r + 3r + 3 = 0$$

$$r^2(r+1) + 2r(r+1) + 3(r+1) = 0$$

$$(r+1)\underbrace{(r^2 + 2r + 3)}_{} = 0$$

$$\Delta = 4 - 12 = -8$$

$$r_{2,3} = \frac{-2 \pm i\,2\sqrt{2}}{2} = -1 \pm i\sqrt{2}$$

correspondingly: $y_2(t) = e^{-t}\cos\sqrt{2}t$

$y_3(t) = e^{-t}\sin\sqrt{2}t$

So:

$$\boxed{y(t) = c_1 e^{-t} + c_2 e^{-t}\cos\sqrt{2}t + c_3 e^{-t}\sin\sqrt{2}t}$$