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1. Determine the Laplace transform $\mathcal{L}\{f\}$ of the following functions:

(a) 5
$$f(t) = \begin{cases} e^{-t} & 0 \le t \le 5 \\ -1 & 5 < t \end{cases}$$

By def. $\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^5 e^{-st} e^{-t}\,dt + \int_5^\infty e^{-st}(-1)\,dt =$

$$= \int_0^5 e^{-(s+1)t}\,dt - \lim_{N\to\infty} \int_5^N e^{-st}\,dt = -\frac{e^{-(s+1)t}}{s+1}\Big|_0^5 + \lim_{N\to\infty}\left(\frac{e^{-st}}{s}\Big|_5^N\right) =$$

$$= -\frac{e^{-(s+1)5}}{s+1} + \frac{1}{s+1} + \lim_{N\to\infty}\left(\frac{e^{-sN}}{s}\right) - \frac{e^{-5s}}{s} =$$

$$= \frac{1-e^{-5(s+1)}}{s+1} - \frac{e^{-5s}}{s}$$ 5

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(b)
$$f(t) = e^{2t} - t^3 + t^2 - \sin 5t$$

$\mathcal{L}\{e^{2t} - t^3 + t^2 - \sin 5t\}(s) = \mathcal{L}\{e^{2t}\}(s) - \mathcal{L}\{t^3\}(s) + \mathcal{L}\{t^2\}(s) - \mathcal{L}\{\sin 5t\}(s) =$

2p.

$$= \frac{1}{s-2} - \frac{3!}{s^4} + \frac{2!}{s^3} - \frac{5}{s^2+5^2} = \frac{1}{s-2} - \frac{6}{s^4} + \frac{2}{s^3} - \frac{5}{s^2+25}$$

2p 2p 2p 2p

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2. Determine the inverse Laplace transform of the given function:

$$\frac{2s^2+3s-1}{(s+1)^2(s+2)}$$

$$\frac{2s^2+3s-1}{(s+1)^2(s+2)} = \frac{A}{(s+1)^2} + \frac{B}{s+1} + \frac{C}{s+2} = \frac{A(s+2)+B(s+1)(s+2)+C(s+1)^2}{(s+1)^2(s+2)}$$

$$\Rightarrow \quad 2s^2+3s-1 = A(s+2)+B(s+1)(s+2)+C(s+1)^2$$

$s=-2$: $\quad 2(-2)^2+3(-2)-1 = C(-2+1)^2 \Rightarrow C=1$ $\quad$ 2p

$s=-1$: $\quad 2(-1)^2+3(-1)-1 = A(-1+2) \Rightarrow A=-2$ $\quad$ 2p

$s=0$: $\quad -1 = 2A+2B+C$ $\quad \Rightarrow B = \frac{(-1-2A-C)}{2} = 1$ $\quad$ 2p

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{2s^2+3s-1}{(s+1)^2(s+2)}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{-2}{(s+1)^2} + \frac{1}{s+1} + \frac{1}{s+2}\right\}(t) = -2te^{-t}+e^{-t}+e^{-2t}$$

4p

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3. Solve the given initial value problem for $y(t)$ using the method of Laplace transforms.

$$y'' + y = u(t-2) \qquad y(0) = 1 \qquad y'(0) = 0$$

$\mathcal{L}\{y'' + y\} = \mathcal{L}\{u(t-2)\}$

$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s$ 2p

$\mathcal{L}\{u(t-2)\} = \frac{e^{-2s}}{s}$

$\Rightarrow \quad s^2Y(s) - s + Y(s) = \frac{e^{-2s}}{s}$

$(s^2+1)Y(s) = \frac{e^{-2s}}{s} + s \quad \Rightarrow Y(s) = \frac{e^{-2s} + s^2}{s(s^2+1)}$ 5p

$\mathcal{L}^{-1}\{Y(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s^2+1)} + \frac{s}{s^2+1}\right\}$

$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s^2+1)}\right\}(t) = f(t-2)u(t-2)$ 2p

$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \quad \Leftrightarrow \quad 1 = A(s^2+1) + (Bs+C)s$

$1 = As^2 + A + Bs^2 + Cs$

$1 = (A+B)s^2 + Cs + A \quad \Rightarrow \quad A = 1$ 2p

$C = 0$ 2p

$B = -1$ 2p

$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s} - \frac{s}{s^2+1}e^{-2s}\right\} = u(t-2) - \cos(t-2)u(t-2)$

$\Rightarrow \mathcal{L}^{-1}\{Y(s)\}(t) = u(t-2) - \cos(t-2)u(t-2) + \cos t$

5p

4. (a) Determine all the singular points of the given equation and classify them as regular or irregular.

(1p)

$$(x^2-4)^2y''+(x-4)y'+xy=0$$

$$y''+\frac{x-4}{(x^2-4)^2}+\frac{x}{(x^2-4)^2}=0$$

$$(\Rightarrow)\quad y''+\frac{x-4}{(x-2)^2(x+2)^2}+\frac{x}{(x-2)^2(x+2)^2}=0 \Rightarrow \pm 2 \text{ singular points (1p)}$$

$x=2$: $xp(x)=\dfrac{x(x-4)}{(x-2)^2(x+2)^2}$ not analytic at $x=2$ (2p) (no need to check $x^2q(x)$)

$x=-2$ $xp(x)=\dfrac{x(x-4)}{(x-2)^2(x+2)^2}$ not analytic at $x=-2$

$\Rightarrow$ $x=\pm 2$ irregular singular points

(b) Find a minimum value for the radius of convergence of a power series solution about x_0.

$$(x+1)y''-3xy'+2y=0 \qquad x_0=1$$

$$y''-\frac{3x}{x+1}y'+\frac{2}{x+1}y=0$$

$x=-1$ is singular point 2p

$d(x_0,x)=2$

$\rho=2$

3p

(c) Determine the convergence set of the given power series

$$\sum_{n=0}^{\infty} \frac{(n+2)!}{n!}(x+2)^n$$

$\lim_{n \to \infty} \left| \frac{(n+3)(n+2)}{(n+2)(n+1)} \right| = 1 \Rightarrow$ radius $= 1$ (2p)

convergence set $= (-2-1, -2+1) = (-3, -1)$ (1p)

check for -3: $\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1)(-1)^n$ not convergent (1p)

for -1 $\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1)$ not convergent (1p)

$\Rightarrow$ Convergence set is the interval open

$\underline{(-3, -1)}$

5. Determine the first three nonzero terms in the Taylor polynomial approximations for the given initial value problem.

$$y''(t) + y(t) = e^t \qquad y(0) = 0 \qquad y'(0) = 0$$

$y(t) = \sum_{n=0}^{\infty} a_n t^n \Rightarrow e^t = 1 + t + \frac{t^2}{2!} + - + \frac{t^n}{n!} + \text{—}$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + \sum_{n=0}^{\infty} a_n t^n = 1 + t + \text{—}$

or $2a_2 + 3\cdot 2 a_3 t + 4\cdot 3 a_4 t^2 + \text{—} + a_0 + a_1 t + a_2 t^2 + - =$ (5p)

$= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \text{—}$ (*)

Note that initial conditions give $a_0 = 0$, $a_1 = 0$

From (*) $\Rightarrow 2a_2 + a_0 = 1 \Rightarrow \boxed{a_2 = \frac{1}{2}}$

$6a_3 + a_1 = 1 \Rightarrow \boxed{a_3 = \frac{1}{6}}$ (5p)

$12a_4 + a_2 = \frac{1}{2} \Rightarrow a_4 = \frac{1}{12}\left(\frac{1}{2} - \frac{1}{2}\right) = 0$

$20a_5 + a_3 = \frac{1}{6} \Rightarrow a_5 = \frac{1}{20}\left(\frac{1}{6} - \frac{1}{6}\right) = \cancel{-\frac{1}{6}\frac{1}{20}} = \cancel{-\frac{1}{120}}$

$30a_6 + a_4 = \frac{1}{4\cdot 3\cdot 2}$

$\Rightarrow \boxed{a_6 = \frac{1}{30}\cdot\frac{1}{24} = \frac{1}{720}}$ (5p)

$\Rightarrow y(t) = \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{720}t^6 + \text{—}$

Or easier:

$$y''(t) = e^t - y(t) \quad , \quad y(0) = 0 \, , \, y'(0) = 0$$

$$\rightarrow y''(0) = 1 - 0 = 1$$

$$y'''(t) = e^t - y'(t) \rightarrow y'''(0) = 1 - 0 = 1$$

$$y^{(4)}(t) = e^t - y''(t) \Rightarrow y^{(4)}(0) = 1 - 1 = 0$$

$$y^{(5)}(t) = e^t - y^{(3)}(t) \Rightarrow y^{(5)}(0) = 1 - 1 = 0$$

$$e^t - y^{(4)}(t) \qquad y^{(6)}(0) = 1$$

10P

$$\rightarrow y(t) = \frac{1}{2!} t^2 + \frac{1}{3!} t^3 + \frac{1}{6!} t^6 + \dots =$$

$$= \frac{1}{2} t^2 + \frac{1}{6} t^3 + \frac{1}{720} t^6 + \dots$$

5P

6. Find a power-series expansion about $x_0 = 0$ for the solution to the given initial value problem. Your answer should include a general formula for the coefficients.

$$y'' + y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k + \sum_{n=0}^{\infty} a_n x^n = 0$$

$\boxed{k \geq 0} \Rightarrow a_{k+2} = -\frac{1}{(k+2)(k+1)} a_n$ 5p

$a_0 = 1$
$a_1 = 0$ (initial conditions)

$\rightarrow a_2 = -\frac{1}{2\cdot 1} a_0$ $\qquad a_3 = -\frac{1}{3\cdot 2} a_1 = 0$

$a_4 = -\frac{1}{4\cdot 3} a_2 = +\frac{1}{4!} a_0$ $\qquad a_5 = -\frac{1}{5\cdot 4} a_3 = 0$

$a_6 = -\frac{1}{6!} a_0$

$a_8 = +\frac{1}{8!} a_0$ $\qquad \boxed{a_{2k+1} = 0 \quad (\forall) k}$ 3p

$\boxed{a_{2k} = (-1)^k \frac{1}{2k!}}$ 5p

So $y(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k!} x^{2k}$ 5p

$\left((-1)^k \rightarrow -2p\right)$