Midterm Examination
due Nov 10 by 6:15pm

This is a take home examination. You have two weeks to work on these problems. Please do not talk with anybody about any of these problems.

(1) All children in Bulgaria are given IQ tests at ages 8 and 16. Let X be the IQ score at age 8 and let Y be the IQ score at age 16 for a randomly chosen Bulgarian 16-year-old. The joint distribution of X and Y can be described as follows. X is normal with mean 100 and standard deviation 15. Given that $X = x$, the conditional distribution of Y is normal with mean $0.8x + 30$ and standard deviation 9.

Among Bulgarian 16-year-olds with $Y = 120$, what fraction have $X \geq 120$?

(2) A circular dartboard has a radius of 1 foot. Tom throws 3 darts at the board until all 3 darts are sticking in the board. The locations of the 3 darts are independent and uniformly distributed on the surface of the board. Let T_1, T_2, and T_3 be the distances from the center to the closest dart, the next closest dart, and the farthest dart, respectively, so that $T_1 \leq T_2 \leq T_3$. Find $\mathbf{E}[T_2]$.

(3) Let $X_1, X_2, \ldots, X_{1000}$ be i.i.d. each taking on both 0 and 1 with probability $\frac{1}{2}$. Put $S_n = X_1 + \cdots + X_n$. Find $\mathbf{E}\left[(S_{1000} - S_{300})\mathbf{1}_{\{S_{700}=400\}}\right]$ and $\mathbf{E}\left[(S_{1000} - S_{300})^2\mathbf{1}_{\{S_{700}=400\}}\right]$.

(4) Find a density function $f(x, y)$ such that if (X, Y) has density f then $X^2 + Y^2$ is uniformly distributed on (0,10).

(5) Let X be a unit exponential random variable (with density $f(x) = e^{-x}, x > 0$) and let Y be an independent $U[0, 1]$ random variable. Find the density of $T = Y/X$.

(6) A box contains 3 balls, numbered 1, 2, 3. Ann randomly chooses a ball and writes down its number (without returning the ball to the box). Call the number X_1. Then Bob randomly chooses a ball and writes

down its number, which we'll call Y_1. The drawn balls are returned to the box. This procedure is done over and over, for a total of 1,000 times, generating random variables $X_1, X_2, \ldots, X_{1,000}$ and $Y_1, Y_2, \ldots, Y_{1,000}$.

Let

$$S = \sum_{1}^{1,000} X_i \quad \text{and} \quad T = \sum_{1}^{1,000} Y_i$$

be Ann's and Bob's respective totals. Find the correlation between S and T.

(7) It is well-known that 23 "random" people have a probability of about 1/2 of having at least 1 shared birthday. There are 365 x 24 x 60 = 525,600 minutes in a year. (We'll ignore leap days.) Suppose each person is labelled by the minute in which the person was born, so that there are 525,600 possible labels. Assume that a "random" person is equally likely to have any of the 525,600 labels, and that different "random" people have independent labels.

  (a) About how many random people are needed to have a probability greater than 1/2 of at least one shared birth-minute? (I want a number.)
  (b) About how many random people are needed to have a probability greater than 1/2 of at least one birth-minute shared by three or more people? (Again, I want a number. You can use heuristic reasoning, but explain your thinking.)