Homework 2
Math 611 Probability

due Monday Oct. 6 2008 by 6:15pm

1. Andre Agassi and Pete Sampras decide to play a number of games together. They play non-stop and at the end it turns out that Sampras won n games while Agassi m where $n > m$. Assume that in fact any possible sequence of games was possible to reach this result. Let $P_{n,m}$ denote the probability that from the first game until the last Sampras is always in the lead. Find:

   (a) $P_{2,1}$; $P_{3,1}$; $P_{n,1}$

   (b) $P_{3,2}$; $P_{4,2}$; $P_{n,2}$

   (c) $P_{4,3}$; $P_{5,3}$; $P_{5,4}$

   (d) Make a conjecture about a formula for $P_{n,m}$.

2. My friend Andrei has designed a system to win at the roulette. He likes to bet on red, but he waits until there have been 6 previous black spins and only then he bets on red. He reasons that the chance of winning is quite large since the probability of 7 consecutive back spins is quite small. What do you think of his system. Calculate the probability the he wins using this strategy.

   Actually, Andrei plays his strategy 4 times and he actually wins three times out of the 4 he played. Calculate what was the probability of the event that just occurred.

3. Let $X_1, X_2, \ldots, X_n$ be independent $U(0,1)$ random variables. Let $M = \max_{1 \le i \le n} X_i$. Calculate the distribution function of M.

4. There is a total of m types of labels on the back of the bottle cap for a certain brand of beer. Collecting all these labeled caps is important since you may win a certain Ferrari model if you have them all. Each newly obtained cap is independent of any of the previous ones.

   (a) Suppose that any of the m types is equally likely. Find the expected number of beers one needs to drink in order to obtain at least one beer cap of each label type.

   (b) Say that the type i beer cap is obtained from a bottle with probability p_i, for all $i = 1, 2, \ldots, m$. Suppose that you drink n total bottles. Find the expectation and variance of the number of distinct labels that you will obtain.

5. We know that the random variables X and Y have joint density $f(x, y)$ (known). Assume that $\mathbf{P}(Y = 0) = 0$. Find the densities of the following variables:

   (a) $X + Y$

   (b) $X - Y$

   (c) XY

   (d) $\frac{X}{Y}$

6. Give a counterexample to the statement $\mathbf{E}(XY) = \mathbf{E}(X)\mathbf{E}(Y)$ implies that X and Y are independent.

7. Ali Baba is caught by the sultan while stealing his daughter. The sultan is being gentle with him and he offers Ali Baba a chance to regain his liberty.

   There are 2 urns and m white balls and n black balls. Ali Baba has to put the balls in the 2 urns however he likes such that no urn is empty. After that the sultan will chose an urn at random then pick a ball from that urn. If the chosen ball is white Ali Baba is free to go, otherwise Ali Baba's head will be at the same level as his legs.

   How should Ali Baba divide the balls to maximize his chance of survival?

As always any exercises assigned in class count for bonus points.