Homework 5
Math 611 Probability

due Monday Nov. 24 2008 by 6:15pm

(1) The joint density of X and Y is given by:

$$f(x,y) = \frac{e^{-x/y}e^{-y}}{y}\ \mathbf{1}_{(0,\infty)\times(0,\infty)}(x,y)$$

Show that $\mathbf{E}[X|Y=y] = y$

(2) The joint density of X and Y is given by:

$$f(x,y) = \frac{e^{-y}}{y}, \quad 0 < x < y,\ 0 < y < \infty$$

Calculate $\mathbf{E}[X^2|Y=y]$

(3) Let $X \sim U[0,1]$. Find $\mathbf{E}[X|X < \frac{1}{2}]$.

(4) Let X be exponential with mean $1/\lambda$, i.e.,

$$f_X(x) = \lambda e^{-\lambda x}\ \mathbf{1}_{(0,\infty)}(x).$$

Find $\mathbf{E}[X|X > 1]$

(5) A prisoner is trapped in a dungeon location that contains 3 doors. The first door leads to a tunnel that takes him to safety after 2 hours of travel. The second door leads to a tunnel that will return him to the same dungeon location after three hours of travel. The third door leads to a tunnel that is returning him to the same dungeon location after five hours of travel. Every door choice is at random and after any choice the doors are magically changed so that the next time it is put in front of the choice the prisoner cannot find his previous choice. What is the expected length of time to exit the dungeon?

(6) In the previous dungeon example let N denote the total number of doors that the prisoner selects before making it to safety. Let T_i denote the travel time corresponding to the i-th choice, $1 \le i \le N$. Let X the total time until reaching safety.

(a) Give an identity that relates X to N and to T_i's.

(b) Find $E[N]$.

(c) Find $E[T_N]$

(d) Find $E[\sum_{i=1}^{N} T_i | N = n]$.

(e) What is $E[X]$?

(7) Also refer to the previous two problems. Assume now that the probabilities of his choice are $0.5, 0.3, 0.2$ respectively for doors 1, 2 and 3. Find the expected number and variance of days until he reaches freedom.

(8) In the same setting as the previous problems, assume that the prisoner chooses at random between doors but there is no magic present and he remembers the previous choice. So, for example if he chose door 2 which returns him to the dungeon he would be really stupid to make the same choice again next time. In this version find the expectation and variance of the number of days to reach safety.

(9) Ann and Bob each attempt 100 basketball free throws. Ann has probability 0.60 of success on each attempt. Bob has probability 0.50 of success on each attempt. The 200 attempts are independent.

What is the approximate numerical probability that Ann and Bob make exactly the same number of free throws?

(10) The random variable whose probability density function is given by:

$$f(x) = \begin{cases} \frac{1}{2}\lambda e^{\lambda x} & , \quad \text{if } x \le 0 \\ \frac{1}{2}\lambda e^{-\lambda x} & , \quad \text{if } x > 0, \end{cases}$$

is said to have a Laplace, sometimes called a *double exponential*, distribution.

(a) Verify that the density above defines a proper probability distribution.

(b) Find the distribution function $F(x)$ for a Laplace random variable.

Now, let X and Y be independent exponential random variables with parameter λ. Let I be independent of X and Y and equally likely to be 1 or -1.

(c) Show that $X - Y$ is a Laplace random variable.

(d) Show that IX is a Laplace random variable.

(e) Show that W is a Laplace random variable where:

$$W = \begin{cases} X & , \quad \text{if } I = 1 \\ -Y & , \quad \text{if } I = -1. \end{cases}$$

(11) Let X, Y, Z be three random variables with joint distribution

$$P(X = k, Y = m, Z = n) = p^3 q^{n-3}$$

for integers k, m, n satisfying $1 \le k < m < n$, where $0 < p < 1$, $p + q = 1$. Find $E\{Z|X, Y\}$.

As always any exercises assigned in class count for bonus points.