Homework 6
Math 611 Probability

This assignment is not going to be graded. It serves as Final test training exercise.

(1) Let $X_1, X_2, \ldots$ be iid Cauchy distributed random variables with density $f(x) = \frac{1}{\pi(1+x^2)}$ and characteristic function $\phi(t) = e^{-|t|}$. Prove that the weak law of large numbers does not hold (i.e., $(X_1 + X_2 + \ldots + X_n)/n$ does not converge to $\mathbf{E}[X_1]$ in probability as $n \to \infty$).

(2) The weak law may hold sometimes even if the mean does not exist. If we dampen the tails of the Cauchy ever so slightly for example by taking the density $f(x) = \frac{c}{(1+x^2)log(1+x^2)}$, show that the weak law of large numbers holds.

(3) Consider the case of the Binomial distribution with $p = 1/2$. Use Stirlings formula: $n! \simeq \sqrt{2\pi n}\, n^n e^{-n}$ to estimate the probability:

$$\sum_{r \geq nx} \binom{n}{r} \frac{1}{2^n}$$

and show that it decays geometrically in n. Can you calculate the geometric ratio

$$\rho(x) = \lim_{n\to\infty} \left[\sum_{r \geq nx} \binom{n}{r} \frac{1}{2^n} \right]^{\frac{1}{n}}$$

explicitly as a function of x for $x > 1/2$?

(4) Prove the inequality $1 - \cos 2t \leq 4(1 - \cos t)$ for all real t. Deduce the inequality $1 - \text{Real } \phi(2t) \leq 4[1 - \text{Real } \phi(t)]$, valid for any characteristic

function. Conclude that if a sequence of characteristic functions converges to 1 in an interval around 0, then it converges to 1 for all real t.

(5) Let $X_1, X_2, \ldots$ be independent random variables each of which is uniformly distributed on $(0,1)$. Let N_n be the number of $X_1, X_2, \ldots, X_n$ which are less than or equal $1/2$. Show that:

$$\sqrt{n}\left(2\frac{N_n}{n} - 1\right) \xrightarrow{d} N(0,1),$$

where $\xrightarrow{d}$ denotes convergence in distribution.

(6) A model for the count of tumors detected in rats exposed to a carcinogen in an experiment assumes:

i. The number M of tumors initiated is a random variable with mean μ and standard deviation σ.

ii. All tumors start at time $t = 0$. The times $T_1, T_2, \ldots, T_M$ at which the tumors are detected are iid random variables with cdf $F(t)$.

Please solve the following problems:

(a) Let J_t be the number of tumors detected by time t. Give the mean and variance of J_t in terms of μ, σ, and F.

(b) Show that if M has the Poisson distribution $\mathcal{P}(\lambda)$ with mean λ, then J_t has the $\mathcal{P}(\lambda F(t))$ distribution.

(7) Let ν be the total number of spots which are obtained in 1000 independent rolls of an unbiased die.

(a) Find $\mathbf{E}[\nu]$

(b) Estimate the probability $\mathbf{P}\,(3450 < \nu < 3550)$

(8) Let S_n be the number of successes in a series of independent trials whose probability of success at the k^{th} trial is p_k. Suppose $p_1, p_2, \ldots, p_n$ depend on n in such a way that:

$$p_1 + p_2 + \cdots + p_n = \lambda, \quad \text{for all } n,$$

while $\max\{p_1, p_2, \ldots, p_n\} \to 0$ when $n \to \infty$. Prove that S_n has a Poisson distribution with parameter λ in the limit as $n \to \infty$.

(9) Let $\{X_i\}_{i \geq 1}$ be a sequence of independent identically distributed random variables with $\mathbf{E}X_1 > 0$. Let $S_n = \sum_{i=1}^n X_i$. Given an $a > 0$, show that $\mathbf{E}\tau < \infty$, where $\tau = \inf\{n \geq 1 : S_n \geq a\}$.

SRW problems:

(10) A compulsive gambler is never satisfied. At each stage he wins \$1 with probability p and loses \$1 otherwise. Find the probability that he is ultimately bankrupt having started with an initial fortune of $\$k$.

(11) Consider a symmetric random walk S with $S_0 = 0$. Let $T = \min\{n \geq 1 : S_n = 0\}$ be the time of the first return of the walk to its starting point. Show that:

$$\mathbf{P}(T = 2n) = \frac{1}{2n-1}\binom{2n}{n}2^{-2n},$$

and deduce that $\mathbf{E}T^\alpha < \infty$ if and only if $\alpha < \frac{1}{2}$.
(Stirling's formula: $n! \simeq \sqrt{2\pi n}\, n^n e^{-n}$).

(12) Consider a symmetric random walk with an absorbing barrier at N and a reflecting barrier at 0 (when the random walk reaches 0 it moves to 1 at the next step with probability one). Let $\alpha_k(j)$ be the probability that the particle having start at k, visits 0 exactly j times before being absorbed at N. By convention, if $k = 0$ then the starting point counts as one visit already. Show that:

$$\alpha_k(j) = \frac{N-k}{N^2}\left(1 - \frac{1}{N}\right)^{j-1} \quad , \quad \forall j \geq 1,\ 0 \leq k \leq N$$

(13) Let $Y(t) = tB(1/t)$, for $t > 0$ and $Y(0) = 0$, with $B(t)$ a standard Brownian motion started at 0.

(a) What is the distribution of Y(t)?
(b) Calculate $Cov(Y(s), Y(t))$ for $s, t \geq 0$.

(c) Argue that $\{Y(t)\}_{t\geq 0}$ is a standard Brownian motion.