# Homework 2

## *Ma623 Stochastic Processes*

due Tuesday Feb 24 2009

These are problems from the Chapter 3 Poisson Process.
Please solve the following problems:
From your textbook do the problem 6 on page 153. Then do the following:

(1) It's the year 2020. Professor Roger Pinkham is now living in Paris. Each weekday, he arrives for lunch at his favorite café at a random time which is uniformly distributed between 12 noon and 12:15 p.m. The time after his arrival until the waiter asks for his order is an independent random variable which is exponential with mean 5 minutes.

a) Given that Professor Pinkham is asked for his order at precisely 12:15 p.m., what is the conditional probability that he arrived before 12:05 p.m.?

b) Given that Professor Pinkham is asked for his order before 12:15 p.m., what is the conditional probability that he arrived before 12:05 p.m.?

(2) Let $X_t, t \geq 0$, be a Poisson process of rate $\lambda$. Let $T_0 = 0$, and let $T_i$ be the time of the $i^{th}$ observation (jump of $X$), if $i \geq 1$. Let

$$N = \inf\{k \geq 1 : T_k - T_{k-1} > 1\}.$$

Find $\mathbf{E}T_N$, $\mathbf{E}N$, and $\mathbf{E}(T_N | N = 8)$.

(3) Buses arrive to a certain stop according to a Poisson process with rate $\lambda$. If you take the bus from that stop then it takes a time $S$ measured from the time you enter the bus to reach home. If you walk from that

bus stop then it takes a time $T$ to reach home. Suppose that the rule you decide to follow is: wait a deterministic amount of time $s$ then if the bus has not arrived yet you walk home.

a) Compute the expected time to reach home from the moment you arrive at the bus stop.

b) Show that if $T < 1/\lambda + S$ then the expectation in part (a) is minimized by letting $s = 0$; and if $T > 1/\lambda + S$ then the expectation in part (a) is minimized by letting $s = \infty$. Also show that when $T = 1/\lambda + S$ all $s$ values give the same expected time.

c) Give an intuitive explanation why it is enough to consider 0 and $\infty$ as possible values for $s$ when minimizing the expected time.

(4) A device recording lightning intensity is attached to a pole during a thunderstorm. Assume that during that time the bolts of lightning occur at that pole according to a Poisson process with rate $\lambda$. However, each bolt registered by the machine renders the device inoperative for a fixed length of time $a$ and it does not register anything that may occur during that interval. Let $R(t)$ denote the number of light-bolts that are registered by the machine at time $t$.

a) Find the probability that the first $k$ bolts of lightning are all registered.

b) For $t \geq (n-1)a$, find $\mathbf{P}\{R(t) \geq n\}$

Hint: Look to type 1 type 2 events for the Poisson process.

(5) A two dimensional Poisson process is a general Poisson process as we described in class with $\mathscr{X} = \mathbb{R}^2$. More specifically for any region of the plane $A$ the number of events in $A$, $N(A)$ is a Poisson random variable with mean $\lambda|A|$, where $|A|$ denotes the area of the region $A$. Furthermore the number of events in any non-overlapping regions are independent. Consider a fixed point (for simplicity the origin) and let $X$ denote the distance from the origin to the nearest event, where distance is measured in the usual euclidian way. Show that:

a) $\mathbf{P}\{X > t\} = e^{-\lambda \pi t^2}$

b) $\mathbf{E}[X] = \frac{1}{2\sqrt{\lambda}}$

c) Now, let $R_i$, denote the distance from the origin to the $i$-th closest event with $i \geq 1$. Show that if we let $R_0 = 0$, then $\pi R_i^2 - \pi R_{i-1}^2$ are iid exponential random variables with rate $\lambda$ for all $i \geq 1$.

In addition do the following simulation exercises:

(I) Using a software package simulate a Poisson process with rate 2 events/min. Using your simulation estimate the probabilities:

$$\mathbf{P}\{N_{[2,4]} = 4\}$$
$$\mathbf{P}\{S_3 \in [3,5]\},$$

where $N_{[2,4]}$ denotes the number of events in the time interval $[2,4]$ minutes, and $S_3$ is the time of the third event.

Calculate what these theoretical probabilities should be and see what was the difference between your simulated probabilities and the theoretical ones for $1,000$, $10,000$ and $100,000$ repetitions respectivelly.

(this is called a *Monte-Carlo* simulation approach).

(II) Using a software package simulate a Poisson process on the plane suitable for problem 5 above. Use $\lambda = 2$. With the help of this simulation answer the following questions:

(a) Estimate the probability that the circle of radius 1 centered in the origin of the plane contains two events.

(b) Estimate the probability in part (a) of the problem 5. Use the origin of the plane as the fixed point, and varying values for the distance $t$ (say $t \in \{0.25, 0.5, 1, 2, 3, 4\}$). Use as many repetitions as you like.

(c) Do the same thing as in part (b) but for the distance $R_2 - R_1$, again with the $R_i$'s defined in the problem 5.

(III) (Bonus) Simulate the inventory problem. Assume that for a year you administer an maritime platform oil facility of the cost of Niger. You have oil tankers arriving according to a Poisson process with rate $0.3/day$. Each oil tanker has a storage capacity that varies depending on the size of the tanker, according to the distribution:

$$f(x) = \frac{1}{1000}(3 - \frac{x}{500})\mathbf{1}_{\{500<x<1500\}},$$

and $x$ is expressed in thousands of barrels.

The oil is loaded in the tankers at a constant rate $10,000$ barrels/hour. Furthermore, the oil wells are linked to the maritime platform and they produce (we will say for simplicity) at a constant rate of $10,000$ barrels/hour. Assume that the facility has infinite storage facility and it works constantly 24 hours 7 days a week. Each time an oil tanker finishes loading, if there is one in the queue it will start loading automatically.

Assume that the cost of oil storage is \$0.5 per thousand of barrels and that each thousand barrels loaded gives you a profit of \$2 find the expected profit after 1 year.

In addition, any problem not mentioned in this assignment and left as an exercise in the class will count as bonus points.