Homework 5

Ma623 Stochastic Processes

due Tuesday April 21 2009

For this assignment please solve the following problems age 73 (ch 2) problem 1 and page 108 (ch 3) problem 2.

(1) Prove that if $X = \{X_n\}_{n=0}^{\infty}$ is a discrete time Markov chain with on the state space $\mathcal{S}$, with initial distribution π_0 and and transition matrix $P = (P_{i,j})$ such that

$$\mathbf{P}\left(X_0 = i_0, \ldots, X_n = i_n\right) = \pi_0(i_0) \prod_{k=0}^{n-1} P_{i_k, i_{k+1}},$$

then the Markov property is true, i.e.:

$$\mathbf{P}\left(X_{n+1} = i_{n+1} | X_0 = i_0, \ldots, X_n = i_n\right) = \mathbf{P}\left(X_{n+1} = i_{n+1} | X_n = i_n\right) = P_{i_n, i_{n+1}}$$

(2) Show that if $P^{(n)}$ is the n-step transition matrix then $P^{(n)} = P^n$ the n-th power of the 1-step transition matrix.

(3) Suppose X is a Markov Chain with transition matrix: $P = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix}$.
Show that $Y_n = (X_n, X_{n+1})$ is also a Markov chain. Find its transition probability and the stationary distribution (if it exists).

(4) Consider the Markov Chain on $\mathcal{S} = \{1, 2, 3, 4\}$ described by:

$$\mathbf{P} = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}$$

a) Determine which states are recurrent and which are transient.

b) Determine the period of each state.

c) If it exists, calculate the stationary distribution of the Markov chain.

(5) Given a Markov Chain on state space $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 8\}$, with transition matrix:

$$\mathbf{P} = \begin{pmatrix} 0.1 & 0 & 0.3 & 0 & 0.6 & 0 & 0 & 0 \\ 0.4 & 0.2 & 0.1 & 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0.3 & 0 & 0 & 0.4 & 0.2 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$

a) Determine which states are positive recurrent, null recurrent, transient.

b) Rearange the matrix so that the recurrent classes are at the beginning and the transient states are all at the end.

c) Given that $X_0 = 1$ calculate the expected number of jumps to transient states before jumping into a recurrent state.

d) Starting at 2 find the probability that the first jump to a recurrent state is to state 7.

e) Calculate the probability that starting at state 3 you eventually reach state 4.

(6) *(SIMULATION PROBLEM). There are many theoretical results for Markov Chains, however in many cases simulation is the most expedient way to study them.*

Suppose there are n people on a Stevens committee discussing the heating issue in the Kidde building. Assume that every time one speaker finishes, one of the other $n-1$ speakers are equally likely to continue the debate. Further, assume that each person speaks an exponential amount of time with parameter λ. How long does it take on average for all the members of the committee to take part in the discussion? Use your choice of λ and n to calculate a numerical answer for this problem. Try to vary the parameters λ and n and see how the answer changes. Does it make sense?