Homework 6

Ma623 Stochastic Processes

due Monday May 11 2009

This is the last assignment. *It consists of three separate parts. It is worth three times as any previous assignments to help the students that need to increase their grade percentage. Only two parts (of your choice) are mandatory, the last remaining part is bonus.*

Continuous time Markov Chain problems:

(I) A pure birth process starting from $X(0) = 0$ has birth parameters $\lambda_0 = 1$, $\lambda_1 = 3$, $\lambda_2 = 2$, and $\lambda_3 = 5$. Find $P_n(t)$ for $n \in \{0, 1, 2, 3\}$.

(II) A factory has 5 machines and one repairman. The operating time until failure of a machine is exponentially distributed with rate 0.2/hour. The repair time of a failed machine is exponentially distributed with rate 0.5/hour. Up to 5 machines may be operating at any given time, the failures being independent of one another. At most one machine may be in repairs at any given time. In the long run what fraction of time is the repairman idle?

(III) Taxis arrive at the pick-up area of a hotel at a Poisson rate μ. Independently passengers arrive at a Poisson rate λ. If there are no passengers waiting, the taxis will wait in a queue, and similarly for the passengers, if there are no taxis at the facility they will form a queue.

For $n, m \geq 0$, let $X(t) = (n, 0)$ if there are n passengers waiting in the queue. Let $X(t) = (0, m)$ if there are m taxis waiting.

Show that $\{X(t)\}_{t\geq 0}$ is a continuous time Markov chain, and write down the balance equations for the states of this Markov chain. You do not need to find limiting probabilities.

(IV) *Birth and death with immigration.* Consider a population of a colonizing species. Suppose that each individual produces offspring at a Poisson rate λ as long as it lives. Moreover suppose new individuals immigrate into the population at a Poisson rate γ.

If the lifetime of the individuals in the population is exponential with mean $\frac{1}{\mu}$, starting with no individuals, find the expected length of time until population size is 3.

Martingale problems:

(V) Consider successive flips of a coin having probability p of l anding heads. Use a martingale argument to compute the expected number of flips until the following sequences appear:

(a) HHTTHHT

(b) HTHTHTH

(VI) Let X be a random variable satisfying

(a) $E[X] \leq m < 0$, and

(b) $P\{-1 \leq X \leq 1\} = 1$.

Suppose $X_1, X_2, \ldots$ are jointly distributed random variables for which the conditional distribution of X_{n+1} given $X_1, X_2, \ldots, X_n$ always satisfies (a) and (b) above. Let $S_n = X_1 + X_2 + \ldots + X_n$, $(S_0 = 0)$ and for $a < x$ let:

$$T_a = \min\{n : x + S_n \leq a\}.$$

Establish the inequality:

$$E[T_a] \leq \frac{(1 + x - a)}{|m|}, \quad a < x.$$

(VII) Let X_n be the total assets of an insurance company at the end of the year n. In each year, n, premiums totalling $b > 0$ are received, and claims A_n are paid, so $X_{n+1} = X_n + b - A_n$. Assume that $A_1, A_2, \ldots$ are i.i.d. normal random variables with mean $\mu < b$ and variance σ^2. The company is ruined if its assets drop to zero or less. Show:

$$P\{\text{ruin}\} \leq e^{-\frac{2(b-\mu)X_0}{\sigma^2}}$$

(VIII) A group of $2n$ people, consisting of n men and n women, are to be independently distributed among m rooms. Each woman chooses room j with probability p_j while each men chooses it with probability q_j, $j \in \{1, 2, \ldots, m\}$. Let X denote the number of rooms that will contain exactly one man and one woman.

(a) Find $\mu = E[X]$

(b) Bound $P\{|X - \mu\} > b\}$ for $b > 0$

(IX) Let $\{X_t\}_{t \geq 0}$ be a continuous time Markov Chain with infinitesimal transition rates q_{ij}, $i \neq j$. Give conditions on q_{ij} so that $\{X_t\}_{t \geq 0}$ becomes a continuous martingale with respect to its standard filtration.

Brownian Motion problems

(X) Let $Y_t = tX_{1/t}$, where X_t is a standard Brownian Motion.

(a) What is the distribution of Y_t?

(b) Compute $Cov(Y_s, Y_t)$.

(c) Argue that Y_t is also a Brownian Motion.

(d) Let $T = \inf\{t > 0 : X_t = 0\}$. Using (c) present an argument that $P\{T = 0\} = 1$.

(XI) Let $\{Z_t\}_{t \geq 0}$ denote a Brownian bridge process (i.e. $Z_t = B_t - tB_1$, where B_t is a standard Brownian motion). Show that if

$$X_t = (t + 1)Z_{\frac{t}{t+1}},$$

then X_t is a Brownian motion.

(XII) Suppose that liquid in a container is placed in a coordinate system, and at time 0, a pollen particle suspended in the liquid is at $(0, 0, 0)$. Let $Z(t)$ be the z-coordinate of the position of the pollen particle after t minutes. Suppose that $\{Z(t)\}_{t \geq 0}$ is a Brownian motion with variance parameter 4. Suppose that after 5 minutes the z-coordinate of the pollen's position is again 0.

(a) What is the probability that after 10 minutes is between -1 and 1?

(b) If after seven minutes the z-coordinate of the pollen position is -2, find the expected value and variance of the z-coordinate of the position after six minutes.

(XIII) Let $\{X(t)\}_{t\geq 0}$ be a Brownian motion with variance parameter σ^2. Show that for all $t \geq 0$, $|X(t)|$ and $\max_{0\leq s\leq t} X(s)$ have the same distribution.

(XIV) *Reflected Brownian Motion:* Suppose that liquid in a container is placed in a coordinate system such that the bottom of the container is placed on the xy-plane. Therefore, whenever a particle reaches the xy-plane it cannot cross the bottom of the container and it is reflected back to the nonnegative side of the z-axis. Suppose that at time zero, a particle is at $(0, 0, 0)$. Let $V(t)$ be the z-coordinate of the particle after t units of time.

Find $\mathbf{E}[V(t)]$, $\mathbf{Var}[V(t)]$, $\mathbf{P}(V(t) \leq z | V(0) = z_0)$

Hint: If $Z(t)$ is a BM with variance parameter σ^2 then:

$$V(t) = \begin{cases} Z(t) & \text{if } Z(t) \geq 0 \\ -Z(t) & \text{if } Z(t) < 0 \end{cases}.$$

$V(t)$ is called *Reflected Brownian motion.*