Final

FE 610 Stochastic Calculus for Financial Engineers

Posted *December 10, 2012*
To be handed in on *December 18 2012*

1. Let us assume that a stock price follows a geometric Brownian motion, where the initial stock price is $S = 530$, and the parameters of the process are $r = 0.01\ (Annualized), \sigma = 0.2\ (Annualized)$. Consider an option chain with 5 strikes: $K_1 = 480, K_2 = 510, K_3 = 530, K_4 = 550, K_5 = 580$, and a single maturity $, T = 1\ month$.
   a) Calculate the European Call and Put option values using the Black-Scholes formula (there is a total of 10 options).
   b) Find an expression using the Black Scholes formula for the options' delta, i.e. $\delta = \frac{\partial C}{\partial S}$ and $\delta = \frac{\partial P}{\partial S}$ where C, P are respectively the prices of call and put options in a).
   c) Calculate the δ for all 10 options.
   d) Verify the put-call parity for all 5 pairs of options

2. For this problem refer to data in problem 1. First construct a binomial tree with 5 steps that approximates the stochastic process described in problem 1. It is up to you to choose the restriction in the tree.
   a) Price the European calls and puts in 1.a) using the tree
   b) Approximate the delta of the options using the tree. To this end use only the options with strike K_3 (both call and put).
   Hint: Recall that $\frac{\partial C}{\partial S}$ is approximately $\frac{C(S+\Delta S)-C(S)}{\Delta S}$
   Compare with the values obtained with the corresponding values in 1.c)
   c) Approximate the American option values with the strike K_3 (both call and put).
   d) Verify the put-call parity for values you obtained in a) and d). Comment.

3. Assume that a process X_t follows the SDE:
$$dX_t = 0.05X_t dt + 0.2X_t dW_t$$
   Use the Ito's lemma to write each of the following processes as Ito processes.
   a) $Y_t = 5X_t$
   b) $Y_t = tX_t^{X_t}$
   c) $Y_t = e^{5t}X_t$

4.

a) Suppose the process X_t solves the following SDE.

$$dX_t = 10(0.02 - X_t)dt + 0.2dW_t, \text{ where } X_0 = \log(100).$$

Give an explicit formula for X_t in terms of W_t and t.

b) Suppose the process X_t solves the following SDE.

$$dX_t = 0.02X_t dt + 0.2X_t dW_t, \text{ where } X_0 = \log(100).$$

As in part a) give a formula for X_t and additionally calculate

$$P(X_{\frac{1}{12}} > \log(103)), \text{ where } t = \frac{1}{12}$$

5. Let the forward rate be modeled as

$$dr_t = (a - br_t)dt + \sigma\sqrt{r_t}dW_t$$

with σ and $a > 0, b \in R$. In this model (CIR model) the price of a zero coupon bond with face value of $1 is given by

$$P(t, T) = A(T - t)e^{-B(T-t)r}$$

Where r is the current rate and

$$A(T - t) = \left[\frac{2\gamma e^{\frac{(a+\gamma)(T-t)}{2}}}{(a + \gamma)(e^{\gamma(T-t)} - 1) + 2\gamma}\right]^{\frac{2a\mu}{\sigma^2}}$$

$$B(T - t) = \frac{2\left(e^{\gamma(T-t)} - 1\right)}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma}$$

$$\gamma = \sqrt{a^2 + 2\sigma^2}$$

We want to price a 3 year fixed/floating interest rate swap (IRS) starting today. To do so we need to calculate R_{swap} (the rate of the fixed payments). Calculation is done such that for the current time, the present value (PV) of future swaps =0.

a) Calculate P(0, 0.5), P(0, 1), …, P(0, 3)

b) Let $P(t, T, s) =$ forward price of a zero coupon bond at t. Express this in term of the zero coupon bond prices. Calculate a formula for the future floating rate on interval (t_k, t_{k+1}) using

$$P(t, t_k, t_{k+1}) = e^{-(t_{k+1} - t_k)f(t, t_k, t_{k+1})}$$

c) Given the formula you calculated in b) and the formula for P(t,T), calculate the PV of all the floating payments.

d) Give a formula depending on R_{swap} for the PV of all the fixed rate future payments

e) From c) and d), calculate the R_{swap} so that the two PV's are matched.