FE 610. Assignment 3

due Monday October 1, 2012 at the beginning of the class (6:15pm).

This assignment is worth 10 points.

1. Problem 2 on page 154 in the Neftci textbook.

2. Let $X_1, X_2, \ldots, X_n$ be independent Uniform random variables on the interval (0,1).

   (a) Let $M = \max_{1 \leq i \leq n} X_i$.
   Calculate the distribution function (CDF) of M.

   (b) Now suppose the sequence is infinite (same type of random variables). Let
   $$S_n = \sum_{i=1}^{n} X_i$$
   Is this S_n a martingale with respect to the sigma algebra generated by the X_i's ($\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$)? Submartingale? Supermartingale? Explain.

   (c) Find a sequence a_n such that $S_n - a_n$ is a martingale.

3. Show that the following M_n processes are martingales

   (a) Let X a random variable with finite expectation. Let $\mathcal{F}_n$ a filtration. Define
   $$M_n = \mathbf{E}[X \mid \mathcal{F}_n].$$
   (the process above is also called a Doob martingale)

(b) Consider $X_1, X_2, \ldots, X_n \ldots$ i.i.d. with $\mathbf{E}[X_1] = 0$. Define:

$$M_n = \sum_{i=1}^{n} X_i.$$

(c) Consider $X_1, X_2, \ldots, X_n \ldots$ i.i.d. with $\mathbf{E}[X_1^2] = \sigma^2$. Define:

$$M_n = \sum_{i=1}^{n} X_i^2 - \sigma^2 n.$$

(d) Consider $X_1, X_2, \ldots, X_n \ldots$ i.i.d. with both $\mathbf{E}[X_1] = 0$ and $\mathbf{E}[X_1^2] = \sigma^2$. Define:

$$M_n = \left(\sum_{i=1}^{n} X_i \right)^2 - \sigma^2 n.$$

(e) Consider $X_1, X_2, \ldots, X_n \ldots$ i.i.d. with $\mathbf{E}[X_1] = 1$. Define:

$$M_n = \prod_{i=1}^{n} X_i.$$

(symbol above means product).

(f) Let X_i be the continuously compounded stock return over the interval $[i-1, i]$. Stock return is defined as $X_i = \log(S_i/S_{i-1})$. First show that the stock value at time n may be calculated as:

$$S_n = S_0 \prod_{i=1}^{n} e^{X_i}$$

Assume that the returns over disjoint intervals are independent and also that $\mathbf{E}[e^{X_i}] = e^r$ (constant). Show that:

$$M_n = e^{-rn} S_n$$

is a martingale