Midterm

FE 610 Stochastic Calculus for Financial Engineers

posted October 12, 2012
to be handed in on October 22, 2012

1. Suppose that $Y_1, Y_2, \ldots$ are independent and identically distributed random variables with $P\{Y_1 = 1\} = p$ and $P\{Y_1 = -1\} = 1 - p$, for some $0 < p < 1/2$. Let

$$S_n = \sum_{i=1}^{n} Y_i$$

denote their partial sums so that $\{S_n\}_n$, $n = 0, 1, 2, \ldots$ is a biased random walk. (Note that S_n is no longer a simple random walk.)

   (a) Is $X_n = S_n - n(2p - 1)$ a martingale?

   (b) Is $M_n = X_n^2 - 4np(1 - p)$ a martingale?

   (c) Is $M_n = \sum_{i=1}^{n} X_i^2 - 4np(1 - p)$ a martingale?

   (d) Is $Z_n = \left(\frac{1-p}{p}\right)^{S_n}$ a martingale?

2. Choose a point A at random in the interval $[0, 1]$ (i.e., the position of A is Uniformly distributed on the interval $[0, 1]$). Let L_1 (respectively L_2) be the length of the bigger (respectively smaller) segment determined by A on $[0, 1]$. Calculate the cumulative distribution function for both these variables, i.e.:

   (a) $F_{L_1}(x) = \mathbf{P}\left(L_1 \leq x\right)$ for $x \in \mathbb{R}$.

(b) $F_{L_2}(x) = \mathbf{P}(L_2 \leq x)$ for $x \in \mathbb{R}$.

3. Give an example of two random variables X and Y such that $\mathbf{E}(XY) = \mathbf{E}(X)\mathbf{E}(Y)$ however X and Y are NOT independent.

4. Let P_t, $t < T$, denote the price of a zero-coupon bond maturing at time $t = T$, with principal $P_t = 1$. Thus P_t is the price of an instrument that pays one dollar at time T and nothing in between (zero coupon). We consider the following model for the evolution of the price of this bond:

$$dP_t = r_t P_t dt + \sigma(T - t)P_t dB_t,$$

where r_t is the short-term interest rate and σ is a known constant. We note that the volatility is $\sigma_t = \sigma(T - t)$.

   (a) Use Itô's lemma to find the SDE for $\log P_t$.

   (b) Find a solution for $\log P_t$. That is, express $\log P_t$ in terms of P_0 and $(r_s, B_s)_{\{0 \leq s \leq t\}}$ only. This type of solution is called a strong solution.

   (c) The yield-to-maturity is defined as

$$R_t = -\frac{\log P_t}{T - t}.$$

   Find the SDE for R_t. Express the volatility (the coefficient of the Brownian motion component) of R_t as a function of $(T - t)$.

   (d) Find the SDE for the discounted bond price $Z_t = P_t/\beta_t$, where $\beta_t = e^{\int_0^t r_s ds}$, and show that it is a martingale.

5. A fair six sided die is rolled. If a 1, 2, or 3 shows up, stop. If not keep rolling until one of those three numbers comes up.

   (a) Let X be a random variable of the result of the die when you stop. What is the sample space and distribution of X?

   (b) Adam has an appointment in the city. He has three options of getting there from Hoboken: the bus, the PATH, or the ferry. Each has its own value of a random variable for time it will take him to get into the city denoted by Y_{bus}, Y_{PATH}, and Y_{ferry} and the distributions are as follows:

- Y_{bus} is exponentially distributed with mean 30 minutes
- Y_{PATH} is uniformly distributed between 15 minutes and 50 minutes
- Y_{ferry} has the distribution $f_{Y_{ferry}}(y) = 2(1-y), 0 < y < 1$ in hours

Adam can't make up his mind as to how to get in the city and decides to roll a die as in the previous part (a) with 1 being for the bus, 2 for the PATH train, and 3 for the ferry.

(c) What is the expected amount of time it takes Adam to get to into the city?

(d) His appointment is in half an hour. What is the probability that he makes it in time assuming that as long as he makes it into the city he makes it.

6. Check whether the following processes are martingales with respect to $\mathscr{F}_t$:

   (a) $X_t = B_t + 4t$

   (b) $X_t = B_t^2$

   (c) $X_t = t^2 B_t - 2\int_0^t sB_s ds$

   (d) $X_t = B_1(t)B_2(t)$, where $B_1(t)$, $B_2(t)$ are independent Brownian motions.

7. Write the following stochastic processes X_t in the standard Itô process form:

$$dX_t = \mu(t,\omega)dt + \sigma(t,\omega)dB_t$$

   for suitable choices of $\mu \in \mathbb{R}^n$, $\sigma \in \mathbb{R}^{n\times m}$ and dimensions n and m:

   (a) $X_t = B_t^2$, where B_t is one dimensional

   (b) $X_t = 1 + t^2 + e^{B_t}$, where B_t is one dimensional

   (c) $X_t = B_1^2(t) + B_2^2(t)$, where (B_1, B_2) is two dimensional

   (d) $X_t = (4 + \sin t, B_t)$, where B_t is one dimensional

   (e) $X_t = (B_1(t)+B_2(t)+B_3(t), B_2(t)-B_1(t)B_3(t))$, where (B_1, B_2, B_3) is three dimensional

8. Consider the following SDE:

$$dY_t = -\frac{Y_t}{1-t}dt + dB_t$$

where $t \in (0,1)$ and $Y_0 = 0$.

Verify that the solution is:

$$Y_t = (1-t)\int_0^t \frac{dB_s}{1-s},$$

and prove that $\lim_{t\to 1} Y_t = 0$ a.s.. This particular process is called the Brownian bridge.

Bonus: Generate and plot 4 different paths of this process.