MA 611 Probability

Final Examination

December 10, 2012

to be emailed or handed in on December 19, 2012 by 6:00pm

(I) Show your work. I may not give credit for a correct answer if you do not show how you got it.

(II) Where possible, give answers as fractions or as decimals to at least 3 significant digits.

(III) **Write neatly and clearly**. Remember, good penmanship is the key to success in life.

You may turn in the exam to either a system admin in Babbio 4th floor (the lab) or you may scan and email the exam to me directly.

(1) Let S_n be the number of successes in a series of independent trials whose probability of success at the k^{th} trial is p_k. Suppose $p_1, p_2, \ldots, p_n$ depend on n in such a way that:

$$p_1 + p_2 + \cdots + p_n = \lambda, \quad \text{for all } n,$$

while $\max\{p_1, p_2, \ldots, p_n\} \to 0$ when $n \to \infty$. Prove that S_n has a Poisson distribution with parameter λ in the limit as $n \to \infty$.

(2) Ann and Bob each attempt 100 basketball free throws. Ann has probability 0.60 of success on each attempt. Bob has probability 0.50 of success on each attempt. The 200 attempts are independent.

What is the approximate numerical probability that Ann and Bob make exactly the same number of free throws?

(3) Let $X_1, X_2, \ldots, X_{100}$ be independent random variables uniformly distributed on the interval $[0, 3]$. Find the approximate numerical probability that $P\left(\prod_{i=1}^{100} X_i \leq 1\right)$. (Here, $\prod_{i=1}^{100} X_i$ means the product of all 100 X_i's).

(4) Let Y be a binomial random variable with parameters n, x where $0 \leq x \leq 1$. For a function $f(\omega)$, continuous on $0 \leq \omega \leq 1$, define:

$$B_n(x, f) = \mathbf{E}\left[f\left(\frac{Y}{n}\right)\right].$$

Show by probabilistic arguments or otherwise that $B_n(x, f) \to f(x)$ as $n \to \infty$ for every x, $0 \leq x \leq 1$.

(5) Let $X_1, X_2, \ldots, X_{100}$ be independent random variables exponentially distributed with mean $\theta = 3$ (i.e. their density is $f(x) = \frac{1}{3}e^{-\frac{x}{3}}$, for $x > 0$). Let $\bar{X}$ be the average of these variables.

(a) Calculate an approximate value for the probability:

$$\mathbf{P}\left(\frac{\bar{X}}{\bar{X} + 17} < 0.5\right)$$

(b) Calculate an approximate value for:

$$\mathbf{P}\left(\frac{\log(\bar{X})}{1+e^{\bar{X}}} < 0.5\right)$$

(6) Let $X_1, X_2, \ldots$ be i.i.d. with density

$$f(x) = \begin{cases} 0 & \text{if } |x| \leq 1 \\ |x|^{-3} & \text{if } |x| > 1. \end{cases}$$

Prove that

$$(n \log n)^{-\frac{1}{2}} \sum_{i=1}^{n} X_i \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$$

and figure out the value of σ^2.

(7) Suppose $X_1, X_2, \ldots$ are Poisson(λ). For all the statistics below:

1. $Y_n = e^{\bar{X}_n} = e^{\frac{1}{n}\sum_{i=1}^{n} X_i}$
2. $Z_n = \left(1 - \frac{1}{n}\right)^{n\bar{X}_n}$
3. $U_n = \frac{1}{\bar{X}_n + \bar{X}_n^2 + \bar{X}_n^3}$

find the limit as $n \to \infty$ (specify the limit type). Furthermore, determine constants μ_Y (respectively μ_Z,μ_U), sequences a_n as well as a limiting random variable Y (resp. Z, U) such that:

$$a_n(Y_n - \mu_Y) \xrightarrow{D} Y,$$

and of course, similarly for the other two.