Midterm Examination
due Oct 23, 2012, by 6:15pm

(1) Suppose you pick two numbers independently at random from $[0, 1]$. Given that their sum is in the interval $[0, 1]$ find the probability that $X^2 + Y^2 < 1/4$.

(2) A man goes to Atlantic City and as part of the trip he receives a special 20$ coupon. This coupon is special in the sense that only the gains may be withdrawn from the slot machine. So the man decides to play a game of guessing red/black. For simplicity let us assume that every time he plays one dollar. The probability of winning is 1/2. Each time he plays the 1$ game he has the same strategy. If he loses 1$ is subtracted from the current coupon amount. If he wins the coupon value remains the same and he immediately withdraws 1$ to count toward the total gains. He plays the game until there are no more funds on the coupon. Find the expected amount of his gains and the expected number of times he plays the 1$ game.

(3) Let $\{X_i\}_{i\geq 1}$ be i.i.d. random variables. Assume that the sums $S_n = \sum_{i=1}^{n} X_i$ have the property $S_n/n \to 0$ almost surely as $n \to \infty$. Show that $\mathbf{E}[|X_1|] < \infty$ and therefore $\mathbf{E}[X_1] = 0$.

(4) The king of Probabilonia has sentenced a criminal to the following punishment. A box initially contains $999,999$ black balls and one white ball. On the day of sentencing, the criminal draws a ball at random. If the ball is white, the punishment is over and the criminal goes free. If the ball is black then two things happen:

(i) the criminal is forced to eat a live toad, and

(ii) the black ball drawn is painted white and returned to the box.

This process is repeated on successive days until the criminal finally draws a white ball. Let X be the number of toads eaten before the punishment ends.

(a) Write down a formula which gives $P\{X = k\}$ exactly.

(b) Estimate the median of X to within three significant digits.

(5) A robot arm solders a component on a motherboard. The arm has small tiny errors when locating the correct place on the board. This exercise tries to determine the magnitude of the error so that we know the physical limitations for the size of the component connections. Let us say that the right place to be soldered is the origin $(0,0)$, and the actual location the arm goes to is (X,Y). We assume that the errors X and Y are independent and have the normal distribution with mean 0 and a certain standard deviation σ.

(a) What is the density function of the distance

$$D = \sqrt{X^2 + Y^2}$$

(b) Calculate its expected value and variance:

$$\mathbf{E}D \text{ and } VarD$$

(c) Calculate

$$\mathbf{E}\left[|X^2 - Y^2|\right]$$

(6) You want to design an experiment where you simulate bacteria living in a certain medium. To this end you know that the lifetime of one bacteria is a random variable X (in hours) distributed with exponential density $\frac{1}{2}e^{-x/2}$. However, you also know that all of these peculiar bacteria live at least 1 hour and die after 10 hours. Thus you need to restrict the generated numbers to the interval $(1, 10)$ by using a conditional density.

(a) Give the exact distribution (or density function) for a random variable which you may use to generate such numbers?

(b) Use any method and write code in any programming language that allows to generate random numbers with this particular conditional density.

(c) Now, suppose in addition that each of the bacteria individual when it dies (and only then) it either divides and creates two new individuals with probability 1/2 or it just dies without any descendants with probability 1/2. Create a program using the lifetime in the previous part that will keep track of the individuals living at any moment in time.

(d) With the previous program start with 1 individual at time $t = 0$ hours and simulate all its offspring until time $t = 240$ hours (10 days). Calculate an approximate value for the expectation of the number of living bacteria 10 days after seeding the culture with 1 bacteria.

(e) Now start with 100 bacteria at time $t = 0$. Simulate each and report the number. Could you use the previous part? Explain.