# Impact of Constraints on the Complexity of Dynamic Spectrum Assignment

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***Abstract*— In this paper we study the complexity of spectrum assignment problems in cognitive radio networks (CRNs) in the presence of several constraints. Although optimal spectrum assignment for secondary transmissions in CRNs is generally believed to be NP complete, the impact of fairness and link quality constraints on the hardness of the problem is not well studied. In this paper we show that when a minimum quality constraint is imposed on secondary transmissions, the spectrum assignment problem can be solved in polynomial time. However, such assignments may not guarantee fairness. We also show that when fairness is desired, even in the presence of quality constraints spectrum assignment problems remain NP complete. We then propose a tree pruning based algorithm to solve distance constrained spectrum assignment problem. We also discuss some heuristic techniques to solve fair distance constrained spectrum assignment problems in polynomial time.**

## I. Introduction

Over the past few years there has been a growing demand for radio resources and at the same time these resources are under utilized due to static spectrum allocation techniques. Dynamic spectrum access (DSA) has been thought of as a solution that would satisfy both the growing demand for radio resources and to efficiently utilize the spectrum. The radio devices that have the capability to dynamically sense the spectrum and access the under utilized bands are called cognitive radios (CR). There are two broad classes of users in cognitive radio networks (CRNs), the primary user is a licensed user of a particular radio frequency band and the secondary users are unlicensed users who cognitively operate without causing harmful interference to the primary user.

Dynamic access technology and smart (cognitive) radios do not always imply efficient utilization of the spectrum. Smarter protocols need to be developed to empower the new technology and optimize the spectrum utilization. The process of optimizing the spectrum utilization by sharing it among bandwidth hungry secondary users is called spectrum management. Spectrum management involves three distinct phases. The first phase is called spectrum scanning, here the spectrum is scanned for secondary usage opportunities. The second phase is spectrum allocation, here the available spectrum is distributed among secondary users for optimal utilization. The third phase is called spectrum handoff, where the secondary user hands off its current spectrum band upon a primary user detection. All the three phases can be managed either by a centralized spectrum server [1], [2] or decentralized [3], [4] and managed by secondary users in a distributed fashion.

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In this paper, we concentrate on the spectrum allocation phase of spectrum management. We also assume the existence of a centralized spectrum server that allocates secondary users the corresponding sub-channels. By sub-channels we mean frequency bands. We study the complexity of an optimal spectrum allocation. The allocation of maximum possible secondary transmissions to a given sub-channel is considered as the optimal solution. We identify two distinct constraints in spectrum assignment: (a) quality constraint (measured by signal to interference noise ratio or distance ratio) and (b) fairness constraint. A formal definition of these constraints are given in later sections. We note that these constraints impact the network capacity, connectivity, lifetime and traffic load. We then study the complexity of optimal spectrum assignment problem in the presence and absence of these constraints. We show that a special class of spectrum assignment problems can be optimally solved in polynomial time in the presence of quality constraint and absence of fairness constraint. This is important in the context of cognitive networks as the set of available spectrum bands is highly time varying resulting in repeated spectrum assignments. Our result shows that in such scenarios and under link quality constraints, the spectrum assignment algorithm scales polynomially with respect to the number of secondary users. Next we show that when fairness is desired, the optimal spectrum assignment problem remains NP complete. Finally, we discuss heuristic algorithms that run in polynomial time to achieve near optimal results for fair spectrum assignments.

The organization of the rest of the paper is as follows, in Section II we model CRNs using a graphs. We then define the quality and fairness constraints for spectrum assignment in Section III. In Section IV we define the spectrum assignment problem followed by its complexity analysis in Section V. An algorithm for distance constrained spectrum assignment is presented in Section VI with simulation results in Section VII. Finally, present some conclusions in Section VIII.

## II. Network Model

To formulate the spectrum assignment problem in a cognitive radio network, we map a given network to a graph

$G$=$(V, E)$, where the vertices, $V$, represent the secondary users (nodes) and the edges, $E$, represent the secondary transmissions (links) between these users. Some of the notations used in the rest of the paper are summarized in Table I.

| | |
|---|---|
| $V$ | Set of all nodes (vertices of G). |
| $e_{i,j}$ | Link formed when node $i$ transmits to node $j$. Note that the existence of $e_{i,j}$ does not imply $e_{j,i}$. |
| $E$ | Set of all possible links in the network (edges in G). |
| $E_t$ | Set of links that are active in sub channel $t$, where $t > 0$. |
| $SIR(e_{i,j}, t)$ | Signal to interference ratio of edge $e_{i,j}$ in the sub channel $t$. |
| $T$ | Total number of active/available sub-channels. |
| $S_i$ | Set of sub-channels assigned to link $e_i$ |

TABLE I
NOTATIONS USED

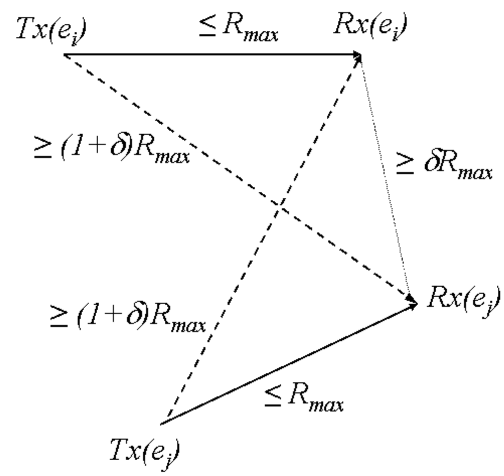


Fig. 1. Pairwise distance constraints between two given links $e_i$ and $e_j$ in a wireless network, with distance ratio constraint of $\delta$.

## III. Spectrum Assignment Constraints

### A. SINR constraint

To assure a minimum quality at each receiver in the network, we define a constraint on the signal to interference plus noise ratio (SINR). The SINR constraint simply says that the assignment should guarantee a desired minimum SINR requirement (say $\gamma_{th}$) for all the links assigned in all of its sub-channels. While the optimal assignment of links to satisfy this constraint requires the consideration of the "physical model" of the system, a simplified analysis can be carried out with a conservative approach leading to the so called "protocol model" [5]. We now define the SINR constraint in the physical model which is then simplified to distance constraint in the protocol model.

Let $E_t$ be the set of all links assigned to the sub channel $t$, then under the physical model the scheduling/assignment is said to satisfy the minimum SINR constraint if,

$$\textit{SINR}(e_{ij}, t) \geq \gamma_{th}, \forall t \in \{1, .., T\}, \ \forall e_{ij} \in E_t \tag{1}$$

### B. Distance constraint

While the physical model simply states the constraint, the protocol model discussed below provides a simplified method to impose the constraint.

If link $e_i$ is assigned sub-channel $t$ (Fig. 1) *i.e.,* $e_i \in E_t$, then it is said to satisfy the minimum *distance ratio constraint* [5], if $\forall e_j \in E_t - e_i$,

$$d(Tx(e_j), Rx(e_i)) \geq (1 + \delta_{th})R_{\max}, \tag{2}$$

where $R_{\max}$ is the maximum transmission radius in the network, $Tx(e_j)$ denotes the transmitter of link $e_j$ and $Rx(e_i)$ is the receiver of link $e_i$. That is, the distance between the receiver of the given link and the transmitters of other active links sharing the same sub channel $t$ should be larger by a factor of $1+\delta_{th}$ compared to the maximum transmission radius of the network.

### C. Fairness

Depending on how many sub-channels each link is assigned to, spectrum assignment can be classified as unfair, 1-fair and fair. A spectrum assignment is called unfair if there is at least one link that is not assigned to any of the sub-channels. This leads to the following definition.

*Definition 1:* let $S_i$ be the set of sub-channels assigned to link $e_i$, then $S_i$ is represented as,

$$S_i = \{t, e_i \in E_t \ \forall t \in \{1, .., T\}\} \tag{3}$$

An assignment is said to be unfair if

$$\exists \ i \in \{1, .., |E|\} \ \textit{s.t.} \ |S_i| = 0 \tag{4}$$

Under such a spectrum assignment the node degree will reduce and the connectivity of the graph may be lost, thus making routing between some source destination pairs impossible. Nevertheless, a network that prefers "maximum overall throughput" regardless of fairness may adopt an unfair policy.

In order to preserve the connectivity of the underlying communication graph, we need every link to be assigned to at least one sub channel. This gives rise to the following definition of 1-fairness.

*Definition 2:* A spectrum assignment is said to satisfy the 1-fairness constraint if,

$$|S_i| \geq 1, \ \forall i \in \{1, .., |E|\} \tag{5}$$

Although 1-fair assignment preserves the connectivity of the graph it can be biased. Some of the links could be assigned more sub channels. Hence, when every link is assigned to exactly same number of sub channels, such an assignment achieves fairness.

*Definition 3:* A spectrum assignment is said to be fair if,

$$|S_i| = d, \ \forall i \in \{1, .., |E|\} \tag{6}$$

where $d \geq 1$ is a constant.

## IV. Spectrum Assignment Problems

*Definition 4:* A spectrum assignment problem is represented as a two-tuple $(\mathbb{G},\mathbb{C})$, where $\mathbb{G}$ is a graph representing the cognitive network and $\mathbb{C}$ is the set of constraints (i.e. distance and fairness).

The spectrum assignment algorithms have two basic objectives:

- maximize the number of links in every sub-channel (maximized total link capacity),
- minimize the total number of sub-channels used subject to all the constraints.

The sub-channels we consider are unused/undersued frequency bands in the spectrum. Depending on fairness and distance constraints we classify spectrum assignment problems as distance constrained spectrum assignment (may be unfair), distance constrained fair assignment and distance constrained 1-fair assignment.

## V. Complexity of Spectrum Assignment Problems

In this section we analyze the complexity of the three spectrum assignment problems. Note that all these problems belong to the class NP [6] in general. However, here we analyze the complexity of special cases of each of these problems when applied in the context of CRNs.

### *A. Distance Constrained Spectrum Assignment*

The distance constrained spectrum assignment problem can be stated as follows.

*Problem 1: Distance constrained spectrum assignment*.

Given a spectrum assignment problem instance $(\mathbb{G},\{\delta_{th}\})$, sub channel $t$ and a positive integer $k$, is there a solution of size $k$ i.e., is there a subset $E_t \subseteq E$ with $|E_t| \geq k$ such that $\delta_{th}$ is satisfied?

To determine the complexity of solving this problem, we explore the implication of distance constraint on NP completeness of the spectrum assignment problem. Our first step in this direction is to arrive at an upper bound on the number of simultaneous transmissions that are possible for a cognitive radio network spread over a geographical area $A$ with distance constraint $\delta_{th}$ and maximum transmission radius $R_{\max}$ (derived from the maximum transmit power). It can be shown by following the work in [5] that under the protocol model the number of simultaneous transmissions, say $J$, is upper bounded by $\frac{4c}{\pi\delta_{th}^2 r_{\max}^2}$, where $0 < c < 1$ is a suitable constant and $r_{\max} = \frac{R_{\max}}{\sqrt{A}}$ is the normalized maximum transmission radius.

*Theorem 1:* For a given distance constraint $\delta_{th}$, radius $r_{\max}$, and area $A$ of the wireless network, the unfair spectrum assignment problem can be solved in polynomial time, if $|E| \gg J$.

*Proof:* To show that this problem is polynomial time solvable, we propose a distance constrained spectrum assignment algorithm *δ-SAA* that solves every instance of the given problem and prove that the algorithm runs in polynomial time.

Note that in Alg. 1 the graph $H = (V_a, E_a)$, vertices of $H$ represent edges of $G$ and any two edges of $G$ that are mutually constrained by the distance constraint $\delta_{th}$ do not have an edge in H between the corresponding vertices. The complexity of generating $H$ is $\Theta(|E|^2)$, which is polynomial in $E$. As none of the vertices in any complete subgraph of $H$ are mutually constrained by any of the constraints specified by $\delta_{th}$, they all can be scheduled in the same sub-channel. This makes every complete subgraph of $H$ a transmission set (set of links that can be assigned the same sub channel) of $G$. Hence the optimal spectrum assignment under no fairness constraint is to determine the maximum clique of $H$. The decision version of this problem is to determine whether a clique of a given size $J$ exists in the graph. Note that,$J \ll |E|$ here is a constant independent of $E$ which implies that $J$ remains constant with respect to an increase in the number of edges $E$. Using this fact we can compute the complexity of *Naive-MC* algorithm as,

$$\textit{Naive-MC}(G,J) \cong \Theta\Big(\sum_{i=1}^{J} \binom{|E|}{i}\Big) \cong \Theta(|E|^J), \forall J \ll |E|$$

The complexity of *δ-SAA* algorithm is $\mathcal{O}(|E|^J)$, which is polynomial in $|E|$. ■

**Algorithm 1** *δ-SAA*

*Input:* $(\mathbb{G},\delta_{th}), r_{\max}$, $t$.
*Output:* $E_t$, set of links assigned to the sub-channel $t$.

```
J = 4c / (π δ_th^2 r_max^2)
if |E| ≫ J then
    H(V_a) ← G(E)
    for v_i = 1 : V_a do
        for v_j = 1 : V_a do
            if Edges v_i and v_j in G are not mutually constrained by the distance constraint δ_th then
                H(E_a) ← H(E_a) ∪ (v_i, v_j)
            end if
        end for
    end for
    MaxClique(V_mc, E_mc) ← Naive-MC(H, J)
    E_t ← MaxClique(V_mc)
end if
```

### *B. 1-Fair Distance Constrained Spectrum Assignment*

The *1-fair* distance constrained spectrum assignment is defined as follows,

*Problem 2: 1-fair distance constrained spectrum assignment*.

*Instance:* A spectrum assignment problem instance $(\mathbb{G},\delta_{th},\textit{1-fair})$ such that $|E| \gg J$ a positive integer $T$, a set of positive integers $\{k_1,..,k_T\}$.

*Question:* Is there a *1-fair distance constrained spectrum assignment* for $\mathbb{G}$ with total number of sub-channels lesser or equal to $T$ and each sub-channel of size $k$? That is, is there a spectrum assignment $\{E_t\}$, with $|\{E_t\}| \leq T$ and $\forall E_{t_i} \in \{E_t\}$, $|E_{t_i}| \geq k_i$ such that all the constraints $\{\delta_{th}, 1-fair)\}$ are satisfied ?

In the following Lemma we show a special class of 1-fair distance constrained spectrum assignment for which, every input instance can be represented as a family $\mathcal{F}$ of feasible transmission sets with the cardinality of $\mathcal{F}$ polynomial in $|E|$. In the related corollary we show that the above transformation can be done in polynomial time.

*Lemma 1:* Given an instance of 1-fair distance constrained spectrum assignment problem in a cognitive radio network with area $A$, maximum radius $r_{\max}$ and distance constraint $\delta_{th}$, the number of feasible transmission sets is polynomial in $|E|$ if $|E| \gg J$.

*Proof:* Follows from Theorem 1 ∎

*Corollary 1:* Transformation $\pi$:*1-fair distance constrained spectrum assignment* $\rightarrow \mathcal{F}$ can be done in polynomial time for a given wireless network with area $A$, maximum radius $r_{\max}$ and constraint $\delta_{th}$ when $|E| \gg J$.

*Proof:* Follows from Theorem 1, using Naive-MC algorithm. ∎

The optimal spectrum assignment $\{E_t\}$ of 1-fair distance constrained spectrum assignment is a collection of transmission sets. Since $\mathcal{F}$ is the set of all feasible transmission sets given the constraints $\{\textit{1-fairness}, \delta_{th}\}$, we have $\{E_t\} \subseteq \mathcal{F}$.

*Lemma 2:* If two sets $s_i, s_j \in \mathcal{F}$ $s.t$ $s_i \subset s_j$ then $s_i \notin \{E_t\}$, where $\{E_t\}$ is the optimal 1-fair distance constrained spectrum assignment solution.

*Proof:* Assume $s_i \in \{E_t\}$. This implies $s_j \in \{E_t\}$, since *1-fair-δ-CAP* outputs sets with maximum cardinality. Since $\{E_t\}$ is optimal, $s_i \notin \{E_t\}$, a contradiction. ∎

We can hence represent the family $\mathcal{F}$ by its maximal feasible sets $\mathcal{F}^{\max}$ such that it is not possible to have two sets $s_i$, $s_j$ such that $s_i \subset s_j$; and $s_i$, $s_j \in \mathcal{F}^{\max}$. Note that any optimal solution on $\mathcal{F}$ will be completely contained in $\mathcal{F}^{\max}$. However the use of $\mathcal{F}^{\max}$ instead of $\mathcal{F}$ may reduce the search complexity.

*Theorem 2: 1-fair distance constrained spectrum assignment problem* is NP-complete.

*Proof:* The maximum cardinality of a transmission set in the family $\mathcal{F}^{\max}$ can be upper bounded by $J$ (from Lemma 1). The objective of *1-fair distance constrained spectrum assignment* will then be to pick a subset $f \subseteq \mathcal{F}$ such that every edge in $E$ is covered by $f$. For $k$ = $J$ and $\{E_t\}$ = $f$, the well known $k$-Set Cover problem and *1-fair distance constrained spectrum assignment problem* are equivalent (from Lemma 1 and Corollary 1). Hence for $J \geq 3$ this problem is NP complete and for $J \leq 2$ optimal solutions can be found in polynomial time using matching techniques [7] [8]. ∎

### C. Fair Distance Constrained Spectrum Assignment

The fair distance constrained spectrum assignment can be defined as follows,

*Problem 3: fair distance constrained spectrum assignment*.

*Instance:* A spectrum assignment problem instance $(\mathbb{G}, \{\textit{fairness}, \delta_{th})$ such that $|E| \gg J$ a positive integer $T$, a set of positive integers $K$=$\{k_1, .., k_T\}$.

*Question:* Is there an *fair-δ-CAP* for $G$ with number of sub-channels lesser or equal to $T$ and each sub-channel of size $K$? That is, is there a spectrum assignment $\{E_t\}$, with $|\{E_t\}| \leq T$, $\forall E_{t_i}, E_{t_j} \in \{E_t\}$, $E_{t_i} \cap E_{t_j} = \phi$ with $|E_{t_i}| \geq k_i$ such that all other constraints are satisfied ?

From Lemma 1 and Corollary 1 it follows that every input instance of fair distance constrained spectrum assignment can be represented as a family $\mathcal{F}$ of feasible transmission sets in polynomial time. The objective of fair distance constrained spectrum assignment is then to find a minimal subcover $f$ which has no overlap. This minimal subcover $f$ will form the optimal spectrum assignment $\{E_t\}$ satisfying all the constraints.

*Theorem 3: fair distance constrained spectrum assignment problem* is NP complete.

*Proof:* For $k$ = $J$ and $\{E_t\}$ = $f$, the SET COVERING II problem [7] is equivalent to *fair distance constrained spectrum assignment*. Hence, for $J \geq 3$ this problem is NP complete and for $J \leq 2$ optimal solutions can be found in polynomial time. ∎

## VI. Algorithms to Solve Distance Constrained Unfair, 1-fair and fair CAP

In this section we propose another algorithm called *Gen-*$\mathcal{F}^{\max}$ which generates the maximal family $\mathcal{F}^{\max}$ using a novel tree pruning approach. For any given *distance spectrum assignment problem* instance, *Gen-*$\mathcal{F}^{\max}$ constructs the corresponding maximal family $\mathcal{F}^{\max}$.

**Algorithm 2** *Gen-*$\mathcal{F}^{\max}$

$i = 1$
$F_i = \{\{1\}\ \{2\}\ ...\ \{|E|\}\}$
$F_{i+1} =$ *Extend-Family*$(F_i, (C))$
**while** $F_{i+1} \neq \phi$ **do**
  $\forall f_i \in F_i; F_i^{max} = \{f_i : f_i \not\subseteq \bigcup_{k=1}^{|F_{i+1}|} F_{i+1}(k)\}$
  $i = i + 1$
  $F_{i+1} =$ *Extend-Family*$(F_i, \mathcal{C})$
**end while**
$F^{max} = \bigcup_i F_i^{max}$

**procedure** $F_{ext} =$ *Extend-Family*$(F_{orig}, \mathcal{C})$

$F_{ext} = \{\phi\}$
**for** $i = 1 : length(F_{orig})$ **do**
  **for** $j = F_{orig}(i, |F_{orig}(i)|) + 1 : |E|$ **do**
    **if** $Satisfy(\{F_{orig}(i)j\}, \mathcal{C})$ **then**
      $F_{ext} = F_{ext} \cup \{F_{orig}(i)j\}$
    **end if**
  **end for**
**end for**

*Theorem 4:* For any given instance of *distance constrained spectrum assignment*, Algorithm *Gen-*$\mathcal{F}^{\max}$ constructs a maximal family $F^{max}$.

*Proof:* Follows by induction on $\bigcup_1^i \mathcal{F}_i$. ∎

The following lemma and theorem establish that the proposed pruning algorithm has polynomial order complexity.

*Lemma 3:* The probability $\alpha$ of co-existence of a pair of links is $1 - \pi\delta_{th}^2 r_{\max}^2$.

*Proof:* The probability of co-existence of any pair of links in the network is given by the probability that the receiving node of the second link is at least $\delta_{th} r_{\max}$ apart from the receiving node of the first link (see Fig. 1). That is, the probability the second receiving node being outside the circle with radius $\delta_{th} r_{\max}$ centered around the first receiving node. Clearly, this probability is equal to, $\alpha = 1 - \pi\delta_{th}^2 r_{\max}^2$. ■

*Theorem 5:* The expected number of feasible transmission sets is upper bounded by $(|E| - \pi\delta^2 R_{\max}^2 n_o)^J$, where $J = \frac{4c}{\pi\delta_{th}^2 r_{\max}^2}$, $r_{\max} = R_{\max}/\sqrt{A}$ and $n_o$ is the edge density in the network.

*Proof:* The expected number of pairs of links in the network is $\alpha\binom{n}{2}$, where $n = |E|$. For any transmission set of cardinality $k$ with $k > 1$, the probability of coexistence is $\alpha^{\binom{k}{2}}$. Therefore, the expected number of feasible transmission sets is given by,

$$E(|\mathcal{F}|) = \sum_{k=2}^{J} \alpha^{\binom{k}{2}} \binom{n}{k} \quad (7)$$

$$= \sum_{k=2}^{J} \alpha^{\frac{k(k-1)}{2}} \frac{n!}{(n-k)!k!} \quad (8)$$

$$\leq \sum_{k=2}^{J} \alpha^{k^2} n^k \quad (9)$$

$$\leq \sum_{k=0}^{J} (\alpha n)^k \quad (10)$$

$$(11)$$

if $(\alpha n) > 1$, $E(|\mathcal{F}|) \cong \mathcal{O}(\frac{(\alpha n)^{J+1}-1}{(\alpha n)-1})$
if $(\alpha n) < 1$, $E(|\mathcal{F}|) \cong \mathcal{O}(\frac{1}{1-(\alpha n)})$
if $(\alpha n) = 1$, $E(|\mathcal{F}|) \cong \mathcal{O}(J)$

The edge density of the network is given by $n_o = |E|/A$. Hence from the above three cases we see that the upper bound of $E(|\mathcal{F}|)$ is, $\mathcal{O}(\frac{(\alpha n)^{J+1}-1}{(\alpha n)-1}) \cong \mathcal{O}(\alpha n)^J = \mathcal{O}(|E| - \pi\delta^2 R_{\max}^2 n_o)^J$. ■

Since $Gen\text{-}\mathcal{F}^{\max}$ looks only at the feasible transmission sets, the running time of $Gen\text{-}\mathcal{F}^{\max}$ is $\mathcal{O}(E(|\mathcal{F}|))$.

### A. Optimal unfair- distance constrained spectrum assignment

An optimal unfair spectrum assignment algorithm is the following. Choose the set with the largest cardinality in $\mathcal{F}^{\max}$ generated by $Gen\text{-}\mathcal{F}^{\max}$. The solution is optimal because, $Gen\text{-}\mathcal{F}^{\max}$ generates the family of all possible large transmission sets. That is, there cannot be a transmission that is larger than the all of the transmission sets in $\mathcal{F}^{\max}$ and satisfy all the specified constraints.

When $|E| \gg J$ this algorithm produces optimal solution for *unfair-CAP* and the running time is same as the $Gen\text{-}\mathcal{F}^{\max}$ algorithm.

### B. Approximate 1-fair distance constrained spectrum assignment

We know that there can be no deterministic algorithm that solves 1-fair distance constrained spectrum assignment problem in polynomial time. Heuristic algorithms [7] that solve minimum cover problem can be used. The input instances of heuristic algorithms that solve minimum cover problem should be mapped from instances of $\{\mathcal{F}^{\max}, E\}$ of 1-fair distance constrained spectrum assignment and the minimum cover output instances should be mapped back as spectrum assignments.

We can use the greedy approximation algorithm discussed in [9]. This algorithm works in a greedy fashion selecting the set that covers the maximum number of uncovered elements in every iteration. The run time complexity of this algorithm is $\mathcal{O}(\sum_{s\in\mathcal{F}^{\max}} |s|)$. It is shown that this algorithm returns a set cover that has an approximation ratio bound of $H(\max\{|s| : s \in \mathcal{F}^{\max}\})$ over the optimal solution, here $H(d)$ denotes the $d^{th}$ harmonic number $H(d) = \sum_{i=1}^{d} 1/i$. From Section V-A we know that $max\{|s| : s \in \mathcal{F}^{\max}\} \leq J$ where $J$ is independent of $E$. Hence the approximation ratio of greedy set cover for this case is $H(J)$.

### C. Approximate fair distance constrained spectrum assignment

By Theorem V-C fair distance constrained spectrum assignment problem is NP complete. The greedy heuristic based on graph coloring [10] can be used directly for the $\delta_{th} = 0$ case. However, for distance constrained assignment a slight modification to the unified algorithm given in [10] generates fair spectrum assignment satisfying the distance constraints.

## VII. Simulation Results

Using simulations, we evaluate the performance of the *unfair*, *1-fair* and *fair* spectrum assignment algorithms for cognitive radio networks with varying secondary user densities. The area of the network used in the simulation is $100 \times 100$ square meters. We fixed the distance ratio constraint $\delta_{th}$ to 3. The maximum transmission range of the network is fixed to $10$ meters. Fig. 2 shows the comparison of the capacities of unfair, 1-fair and fair spectrum assignment algorithms as a function of average number of edges in the network. We can see that as the fairness constraint is relaxed from exact fairness to unfair the capacity achieved increases. For example, Fig. 2 shows that for a network with $80$ edges, there is a $100\%$ gain in capacity when the fairness is relaxed from exact fairness to 1-fairness and $25\%$ gain when fairness constraint is relaxed from 1-fairness to unfair.

To study the impact of distance constraint $\delta_{th}$ on the capacity of the spectrum assignment, we fix the number of edges in the network to $24$ and vary the distance constraint from $0$ to $10$. Fig 3 plots the reduction in capacity due to increase in the distance constraint ($\delta_{th}$). Here we can observe that for smaller distance constraints 1-fair and unfair CAA perform comparably, however as the distance constraint is increased 1-fair CAA performs poorly compared to unfair CAA. We can also observe that exact fair spectrum assignment

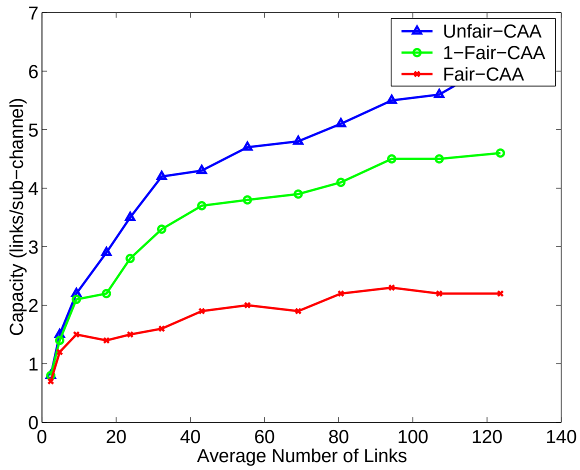


Fig. 2. Performance of unfair, 1-fair, and fair CAA in terms of average number of links per sub-channel versus the number of edges, in a network with area $100 \times 100$ square meters, distance constraint $\delta_{th} = 3$ and maximum range = 10 meters.

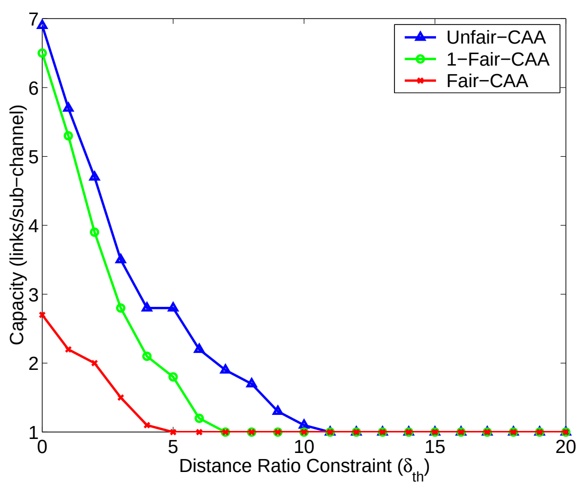


Fig. 3. Performance of unfair, 1-fair, and fair CAA in terms of average number of links per sub-channel versus the distance constraint $\delta_{th}$, in a network with 24 edges, area $100 \times 100$ square meters and maximum range = 10 meters.

quickly converges to the one link per sub-channel worst case behavior compared to 1-fair and unfair assignments.

We also study the capacity of the assignment for networks with different maximum transmission range, $R_{max}$. We fix $\delta_{th} = 3$, number of nodes to 30 and vary the maximum allowable transmission range from 1 to 20 meters. Fig. 4 plots the capacity versus increase in the maximum allowable range. We can observe that the capacities of spectrum assignments increase up to a point and then decreases. This is because at lower transmission range, the network has very few edges and many disconnected nodes. As the transmission range increases the number of edges and hence the capacity also increases and reaches the optimum. The decrease in capacity can be explained by the bound $J$ where $r_{\max}$ appears in the denominator.

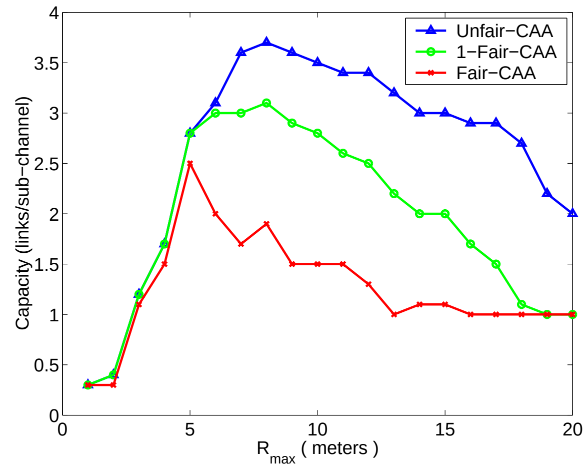


Fig. 4. Performance of unfair, 1-fair, and fair CAA in terms of average number of links per sub-channel versus the maximum range, in a network with 30 nodes, area $100 \times 100$ square meters and $\delta_{th} = 3$.

## VIII. Conclusions

In this paper we systematically studied the spectrum assignment problem in presence of fairness and quality constraints. We showed that a sub-class of spectrum assignment problems in the presence of distance constraint can be optimally solved in polynomial time. We also showed that with fair and 1-fair constraints the spectrum assignment problem remains NP complete. A novel tree pruning based spectrum assignment algorithm was presented and applied to solve distance constrained spectrum allocation in polynomial time. Existing heuristic algorithms for approximate 1-fair and fair spectrum assignment were discussed.

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