Exam I Solutions for Ma 221 2004 Fall.

1 Exam IA

In Problems $1-3$ solve the equations:

1 [25 **pts.**]

$$y' - y\tan t = \sin t \quad y(0) = 2$$

Note: $\int \tan t dt = -\ln(\cos t) + C$.
Solution: This equation is first order linear with $P(t) = -\tan t$. Hence

$$e^{\int P(t)dt} = e^{-\int \tan t dt} = e^{\ln(\cos t)} = \cos t$$

Multiplying the original DE by $\cos t$ we get

$$\cos t y' - y\sin t = \cos t \sin t$$

or

$$(y\cos t)' = \cos t \sin t$$

so that integrating both sides leads to

$$y\cos t = \int \cos t \sin t dt = \frac{\sin^2 t}{2} + C \;\text{ or }\; -\frac{\cos^2 t}{2} + C'$$

so

$$y(t) = \frac{\sin^2 t}{2\cos t} + \frac{C}{\cos t} \;\text{ or }\; y(t) = -\frac{\cos t}{2} + \frac{C'}{\cos t}$$

The initial condition implies

$$C = 2 \;\text{ or }\; C' = \frac{5}{2}$$

Thus

$$y(t) = \frac{\sin^2 t}{2\cos t} + \frac{2}{\cos t} \quad \text{or} \quad y(t) = -\frac{\cos t}{2} + \frac{5}{2\cos t}$$

These two expressions are equivalent since

$$y(t) = \frac{\sin^2 t}{2\cos t} + \frac{2}{\cos t} = \frac{1-\cos^2 t}{2\cos t} + \frac{2}{\cos t} = -\frac{\cos t}{2} + \frac{5}{2\cos t}$$

2 [25 **pts.**]

$$\frac{dy}{dx} = -\frac{3x^2y^2 + 2xy}{2x^3y + x^2 + y}$$

Solution: We rewrite the equation as

$$\left(3x^2y^2 + 2xy\right) dx + \left(2x^3y + x^2 + y\right) dy = 0$$

Letting $M = 3x^2y^2 + 2xy$ and $N = 2x^3y + x^2 + y$, we see that $M_y = 6x^2y + 2x = N_x$ so this equation is exact. The there exists a function $F(x, y)$ such that

$$\begin{aligned} F_x &= M = 3x^2y^2 + 2xy \\ F_y &= N = 2x^3y + x^2 + y \end{aligned}$$

Integrating the first equation with respect to x while holding y fixed leads to

$$F = x^3y^2 + x^2y + g(y)$$

so

$$F_y = 2x^3y + x^2 + g'(y) = N = 2x^3y + x^2 + y$$

Therefore $g'(y) = y$, so that $g(y) = \frac{y^2}{2} + C$. Hence

$$F = x^3y^2 + x^2y + \frac{y^2}{2} + C$$

and the solution is given by

$$x^3y^2 + x^2y + \frac{y^2}{2} = K$$

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3 [25 **points**]

$$\frac{dx}{dt} - tx = t^3x^2$$

Note: $\int t^3 e^{\frac{t^2}{2}}dt = t^2e^{\frac{1}{2}t^2} - 2e^{\frac{1}{2}t^2} + C$
Solution: This is a Bernoulli equation. The equation may be rewritten as

$$x^{-2}x' - tx^{-1} = t^3$$

Let $v = x^{-1}$. Then $v' = -x^{-2}x'$ and the DE can be written as

$$-v' - tv = t^3$$

or

$$v' + tv = -t^3$$

Then

$$e^{\int Pdt} = e^{\int tdt} = e^{\frac{t^2}{2}}$$

Multiplying the DE by this we get

$$e^{\frac{t^2}{2}}v' + te^{\frac{t^2}{2}}v = -t^3e^{\frac{t^2}{2}}$$

or

$$\left(e^{\frac{t^2}{2}}v\right)' = -t^3e^{\frac{t^2}{2}}$$

Integrating gives

$$e^{\frac{t^2}{2}}v = -t^2e^{\frac{1}{2}t^2} + 2e^{\frac{1}{2}t^2} + C$$

so that

$$\frac{1}{x} = -t^2 + 2 + Ce^{-\frac{t^2}{2}}$$

4 a (15 **pts.**) The differential equation

$$y'' - 9y = 2e^{3x} \tag{(*)}$$

has the general solution

$$y(x) = c_1e^{-3x} + c_2e^{3x} + \frac{1}{3}xe^{3x}$$

Find the solution or solutions (if they exist) to $(*)$ with the initial conditions $y(0) = -\frac{1}{9}, y'(0) = \frac{1}{3}$.

Solution:

$$y(0) = c_1 + c_2 = -\frac{1}{9}$$

$$y'(x) = -3c_1e^{-3x} + 3c_2e^{3x} + \frac{1}{3}e^{3x} + xe^{3x}$$

$$y'(0) = -3c_1 + 3c_2 + \frac{1}{3} = \frac{1}{3}$$

or

$$-c_1 + c_2 = 0$$

so $c_1 = c_2$ and $2c_1 = -\frac{1}{9}$ so $c_1 = c_2 = -\frac{1}{18}$. Thus

$$y(x) = \frac{1}{3}xe^{3x} - \frac{1}{18}e^{3x} - \frac{1}{18}e^{-3x}$$

4b (10 **pts.**) Solve the equation

$$x\frac{dy}{dx} = 2(y-4) \quad y \neq 4$$

Solution: This equation is separable. Hence

$$\frac{dy}{y-4} = 2\frac{dx}{x}$$

Therefore

$$\ln(y-4) = 2\ln x + C$$

or

$$\frac{y-4}{x^2} = K$$

2 Exam IB

In Problems $1-3$ solve the equations:

1 [25 **pts.**]

$$\begin{aligned} y' + y\cot t &= \cos t \\ y\left(\frac{\pi}{2}\right) &= 3 \end{aligned}$$

Note: $\int \cot t dt = \ln(\sin t) + C$.

Solution: This equation is first order linear with $P(t) = \cot t$. Hence the integrating factor is

$$e^{\int P(t)dt} = e^{\int \cot t dt} = e^{\ln(\sin t)} = \sin t$$

Multiplying the orginal DE by $\cos t$ we get

$$\sin t y' + y \cos x = \sin t \cos t$$

or

$$(y \sin t)' = \sin t \cos t$$

so that integrating both sides leads to

$$y \sin t = \int \sin t \cos t dt = \frac{\sin^2 t}{2} + C \quad \text{or} \quad -\frac{\cos^2 t}{2} + C'$$

so

$$y(t) = \frac{\sin t}{2} + \frac{C}{\sin t} \quad \text{or} \quad y(t) = -\frac{\cos^2 t}{2 \sin t} + \frac{C'}{\sin t}$$

The initial condition implies

$$C = \frac{5}{2} \quad \text{or} \quad C' = 3$$

Thus

$$y(t) = \frac{\sin t}{2} + \frac{5}{2 \sin t} \quad \text{or} \quad y(t) = -\frac{\cos^2 t}{2 \sin t} + \frac{3}{\sin t}$$

These two expressions are equivalent since

$$y(t) = \frac{\sin^2 t}{2 \cos t} + \frac{2}{\cos t} = \frac{1 - \cos^2 t}{2 \cos t} + \frac{2}{\cos t} = -\frac{\cos t}{2} + \frac{5}{2 \cos t}$$

2 [25 **pts.**]

$$\frac{dy}{dx} = -\frac{4xy^2 + 6xy}{4x^2y + 3x^2 + 2}$$

Solution: We rewrite the equation as

$$\left(4xy^2 + 6xy\right) dx + \left(4x^2y + 3x^2 + 2\right) dy = 0$$

Letting $M = 4xy^2 + 6xy$ and $N = 4x^2y + 3x^2 + 2$, we see that $M_y = 8xy + 6x = N_x$ so this equation is exact. The there exists a function $F(x, y)$ such that

$$\begin{aligned} F_x &= M = 4xy^2 + 6xy \\ F_y &= N = 4x^2y + 3x^2 + 2 \end{aligned}$$

Integrating the first equation with respect to x while holding y fixed leads to

$$F = 2x^2y^2 + 3x^2y + g(y)$$

so

$$F_y = 4x^2y + 3x^2 + g'(y) = N = 4x^2y + 3x^2 + 2$$

Therefore $g'(y) = 2$, so that $g(y) = 2y$. Hence

$$F = 2x^2y^2 + 3x^2y + 2y$$

and the solution is given by

$$2x^2y^2 + 3x^2y + 2y = C$$

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3 [25 **points**]

$$\frac{dx}{dt} - \frac{1}{t}x = 3t^2x^3$$

Solution: We observe that the problem is a Bernoulli d.e. and first rewrite the equation as

$$x^{-3}x' - \frac{1}{t}x^{-2} = 3t^2$$

Let $v = x^{-2}$. Then $v' = -2x^{-3}x'$ and the DE can be written as

$$-\frac{1}{2}v' - \frac{1}{t}v = 3t^2$$

or

$$v' + \frac{2}{t}v = -6t^2$$

Then

$$e^{\int P dt} = e^{\int \frac{2}{t} dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

Multiplying the DE by this we get

$$t^2 v' + 2tv = -6t^4$$

or

$$\left(t^2 v\right)' = -6t^4$$

Integrating gives

$$t^2 v = -\frac{6}{5} t^5 + C$$

so that

$$\frac{1}{x^2} = -\frac{6}{5} t^3 + C t^{-2}$$

4 a (15 **pts.**) The differential equation

$$y'' - 9y = 9x^2 \tag{(*)}$$

has the general solution

$$y(x) = c_1 e^{-3x} + c_2 e^{3x} - x^2 - \frac{2}{9}$$

Find the solution or solutions (if they exist) to $(*)$ with the initial conditions $y(0) = -\frac{1}{9}, y'(0) = \frac{1}{3}$.

Solution:

$$y(0) = c_1 + c_2 - \frac{2}{9} = -\frac{1}{9}$$

$$y'(x) = -3c_1 e^{-3x} + 3c_2 e^{3x} + 2x$$

$$y'(0) = -3c_1 + 3c_2 = \frac{1}{3}$$

Thus the equations for the constants are

$$\begin{aligned} c_1 + c_2 &= \tfrac{1}{9} \\ -c_1 + c_2 &= \tfrac{1}{9} \end{aligned}$$

Adding these yields $2c_2 = \frac{2}{9}$ or $c_2 = \frac{1}{9}$ and thus $c_1 = 0$.

$$y(x) = \frac{1}{9} c_2 e^{3x} - x^2 - \frac{2}{9}$$

4b (10 **pts.**) Solve the equation

$$x^2 \frac{dy}{dx} = 2\,(y+4) \qquad y \neq -4$$

Solution: This equation is separable. Hence

$$\frac{dy}{y+4} = \frac{2}{x^2} dx$$

Therefore

$$\ln|y+4| = -\frac{2}{x} + C$$