Exam II Solutions for Ma 221 2004 Fall.

1 Exam IIA

1. Consider the differential equation

$$y'' - 3y' - 4y = 3e^{2x} + 2\sin x - 8e^{-x}$$

a (10 pts.) Find the homogeneous solution of this equation.

Solution: The characteristic equation is

$$p(r) = r^2 - 3r - 4 = 0$$

This can be factored into

$$(r-4)(r+1) = 0$$

so $r = 4, -1$. The homogeneous solution is therefore

$$y_h = C_1 e^{4x} + C_2 e^{-x}$$

b (25 pts.) Find a particular solution of this equation.

Solution: We find a particular solution corresponding to each term on the right hand side and then add them together.

$3e^{2x}$: Since e^{2x} is not a homogeneous solution and therefore $p(2) \neq 0$, then

$$y_{p_1} = \frac{3e^{2x}}{p(2)} = \frac{3e^{2x}}{-6} = -\frac{1}{2}e^{2x}$$

$2\sin x$: We consider the equation

$$y'' - 3y' - 4y = 2\sin x$$

and the companion equation

$$v'' - 3v' - 4v = 2\cos x$$

Multiplying the first equation by i, adding the two equations and letting $w = v + iy$ leads to

$$w'' - 3w' - 4w = 2(\cos x + i\sin x) = 2e^{ix}$$

Since $p(i) = (i)^2 - 3i - 4 = -5 - 3i$

$$w_p = -\frac{2e^{ix}}{5+3i}$$

We need the imaginary part of w_p, which will be y_{p_2}. Now

$$w_p = -\frac{2e^{ix}}{5+3i} \times \frac{5-3i}{5-3i} = -\frac{2(5-3i)}{34}e^{ix} = -\frac{1}{17}(5-3i)(\cos x + i\sin x)$$

Thus

$$y_{p_2} = \frac{1}{17}(3\cos x - 5\sin x)$$

Alternative solution for y_{p_2} : Let $y_{p_2} = A\cos x + B\sin x$. Differentiating twice and plugging into the DE leads to

$$-A\cos x - B\sin x - 3(-A\sin x + B\cos x) - 4(A\cos x + B\sin x) = 2\sin x$$

This implies

$$\begin{aligned} 3A - 5B &= 2 \\ -5A - 3B &= 0, \end{aligned}$$

Solution is: $\left\{A = \frac{3}{17}, B = -\frac{5}{17}\right\}$ which leads to the same y_{p_2}.

$-8e^{-x}$: Since $p(-1) = 0$ and $p'(r) = 2r - 3$ so that $p'(-1) = -5 \neq 0$ we have that

$$y_{p_3} = -\frac{8xe^{-x}}{-5} = \frac{8}{5}xe^{-x}$$

c (10 pts.) Give a general solution of this equation.

$$y = y_h + y_{p_1} + y_{p_2} + y_{p_3} = C_1 e^{4x} + C_2 e^{-x} - \frac{1}{2} e^{2x} + \frac{1}{17}(3\cos x - 5\sin x) + \frac{8}{5} x e^{-x}$$

2 Consider the differential equation

$$x^2 y'' - 4xy' + 6y = x^2 \ln x \quad x > 0 \qquad ((*))$$

2 a (10 pts.) Find two linearly independent solutions of the homogeneous equation corresponding to $(*)$.

Solution: This is an Euler equation. The homogeneous equation is

$$x^2 y'' - 4xy' + 6y = 0$$

so that $p = -4$ and $q = 6$. The equation for m is $m^2 + (p-1)m + q = 0$ or in this case

$$m^2 - 5m + 6 = (m-2)(m-3) = 0$$

The two linearly independent solutions are x^2 and x^3.

2 b (10 pts.) Find the value of Wronskian of the two linearly independent solutions you found in 2a.

Solution:

$$W\left[x^2, x^3\right] = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4$$

2c (25 **pts.)** Find a particular solution to $(*)$.

Note:

$$\begin{aligned} \int \frac{(\ln x)^n}{x} dx &= \frac{(\ln x)^{n+1}}{n+1} + C \\ \int \frac{\ln x}{x^n} dx &= -\left[\frac{\ln x}{(n-1)x^{n-1}} + \frac{1}{(n-1)^2 x^{n-1}}\right] + C \qquad n \neq 0, 1 \end{aligned}$$

Solution: We use variation of parameters. Let

$$y_p = v_1 y_1 + v_2 y_2$$

where $y_1 = x^2$ and $y_2 = x^3$. The two equations for v_1' and v_2' are

$$\begin{aligned} v_1' y_1 + v_2' y_2 &= 0 \\ v_1' y_1' + v_2' y_2' &= \frac{f(x)}{a(x)} \end{aligned}$$

Since for our equation $f(x) = x^2 \ln x$ and $a(x) = x^2$ these two equations become

$$\begin{aligned} v_1' x^2 + v_2' x^3 &= 0 \\ 2v_1' x + v_2' \left(3x^2\right) &= \frac{x^2 \ln x}{x^2} = \ln x \end{aligned}$$

Then

$$v_1' = \frac{\begin{vmatrix} 0 & x^3 \\ \ln x & 3x^2 \end{vmatrix}}{\begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}} = -\frac{\ln x}{x}$$

and

$$v_2' = \frac{\begin{vmatrix} x^2 & 0 \\ 2x & \ln x \end{vmatrix}}{\begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}} = \frac{\ln x}{x^2}$$

$$\begin{aligned} \int \frac{(\ln x)^n}{x} dx &= \frac{(\ln x)^{n+1}}{n+1} + C \\ \int \frac{\ln x}{x^n} dx &= -\left[\frac{\ln x}{(n-1)\, x^{n-1}} + \frac{1}{(n-1)^2\, x^{n-1}}\right] + C \qquad n \neq 0, 1 \end{aligned}$$

Thus

$$\begin{aligned} v_1 &= -\frac{(\ln x)^2}{2} \\ v_2 &= -\left[\frac{\ln x}{x} + \frac{1}{x}\right] \end{aligned}$$

and

$$y_p = -x^2 \frac{(\ln x)^2}{2} - x^3 \left[\frac{\ln x}{x} + \frac{1}{x} \right]$$

Note: One need not include the $-x^2$ term in the above, since x^2 is a homogeneous solution.

2d (10 **pts.**) Give a general solution to $(*)$.

$$y = y_h + y_p = C_1 x^2 + C_2 x^3 - x^2 \frac{(\ln x)^2}{2} - x^2 \ln x$$

2 Exam IIB

1. Consider the differential equation

$$y'' + 4y' - 5y = 3e^{2x} + 2\cos x - 8e^x$$

a (10 pts.) Find the homogeneous solution of this equation.

Solution: The characteristic equation is

$$p(r) = r^2 + 4r - 5 = 0$$

This can be factored into

$$(r-1)(r+5) = 0$$

so $r = 4, -1$. The homogeneous solution is therefore

$$y_h = C_1 e^x + C_2 e^{-5x}$$

b (25 pts.) Find a particular solution of this equation.

Solution: We find a particular solution corresponding to each term on the right hand side and then add them together.

$3e^{2x}$: Since e^{2x} is not a homogeneous solution and therefore $p(2) \neq 0$, then

$$y_{p_1} = \frac{3e^{2x}}{p(2)} = \frac{3e^{2x}}{7} = \frac{3}{7} e^{2x}$$

$2\cos x$: We consider the equation

$$u'' + 4u' - 5u = 2\cos x$$

and the companion equation

$$v'' + 4v' - 5v = 2\sin x$$

Multiplying the first equation by i, adding the two equations and letting $w = u + iv$ leads to

$$w'' + 4w' - 5w = 2(\cos x + i\sin x) = 2e^{ix}$$

Since $p(i) = (i)^2 + 4i - 5 = -6 + 4i$

$$w_p = -\frac{2e^{ix}}{6 - 4i}$$

We need the real part of w_p, u, which will be y_{p_2}. Now

$$w_p = -\frac{2e^{ix}}{6-4i} \times \frac{6+4i}{6+4i} = -\frac{2(6+4i)}{52}e^{ix} = -\frac{1}{13}(3+2i)(\cos x + i\sin x)$$

Thus

$$y_{p_2} = -\frac{1}{13}(3\cos x - 2\sin x)$$

Alternative solution for y_{p_2} : Let $y_{p_2} = A\cos x + B\sin x$. Differentiating twice and plugging into the DE leads to

$$-A\cos x - B\sin x + 4(-A\sin x + B\cos x) - 5(A\cos x + B\sin x) = 2\cos x$$

This implies

$$\begin{aligned} -6A + 4B &= 2 \\ -4A - 6B &= 0 \end{aligned}$$

, Solution is: $\left[A = -\frac{3}{13}, B = \frac{2}{13}\right]$ which leads to the same y_{p_2}.

$-8e^x$: Since $p(1) = 0$ and $p'(r) = 2r + 4$ so that $p'(1) = 6 \neq 0$ we have that

$$y_{p_3} = \frac{8xe^x}{6} = \frac{4}{3}xe^x$$

c (10 **pts.)** Give a general solution of this equation.

$$y = y_h + y_{p_1} + y_{p_2} + y_{p_3} = C_1 e^x + C_2 e^{-5x} + \frac{3}{7} e^{2x} + \frac{1}{13} (-3\cos x + 2\sin x) + \frac{4}{3} x e^x$$

2 Consider the differential equation

$$x^2 y'' - 3xy' + 3y = x^3 \ln x \qquad x > 0 \tag{(*)}$$

2 a (10 pts.) Find two linearly independent solutions of the homogeneous equation corresponding to $(*)$.

Solution: This is an Euler equation. The homogeneous equation is

$$x^2 y'' - 3xy' + 3y = 0$$

so that substitution of $y = x^m$ and simplifying gives

$$m^2 - 4m + 3 = (m-1)(m-3) = 0$$

The two linearly independent solutions are x and x^3.

2 b (10 pts.) Find the value of Wronskian of the two linearly independent solutions you found in 2a.

Solution:

$$W\left[x^2, x^3\right] = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3.$$

2c (25 **pts.)** Find a particular solution to $(*)$.

Note:

$$\begin{aligned} \int \frac{(\ln x)^n}{x} dx &= \frac{(\ln x)^{n+1}}{n+1} + C \\ \int x^n \ln x dx &= \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1} + C \end{aligned}$$

Solution: We use variation of parameters. Let

$$y_p = v_1 y_1 + v_2 y_2$$

where $y_1 = x$ and $y_2 = x^3$. Then

$$\begin{aligned} y_p &= x v_1 + x^3 v_2 \\ y_p' &= v_1 + 3x^2 v_2 + x v_1' + x^3 v_2' \end{aligned}$$

To simplify the next step and obtain the second equation, we set

$$x v_1' + x^3 v_2' = 0.$$

Then

$$y_p'' = 6 v_2 + v_1' + 3x^2 v_2.$$

Substitution into the d.e. produces our two equations for v_1 and v_2.

$$\begin{aligned} x v_1' + x^3 v_2' &= 0 \\ x^2 v_1' + 3x^4 v_2' &= x^3 \ln x \end{aligned}$$

Then multiplying the first equation by x and subtracting it from the second yields

$$2x^4 v_2' = x^3 \ln x$$

or

$$v_2' = \frac{1}{2x} \ln x.$$

Subtituting this into the first equation, we have

$$v_1' = -\frac{1}{2} x \ln x.$$

Using the integrals provided, we have

$$\begin{aligned} v_1 &= -\int x \ln x dx = -\frac{1}{2}\left(\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right) \\ v_2 &= \int \frac{1}{x} \ln x dx = \frac{1}{4}(\ln x)^2 \end{aligned}$$

and

$$y_p = -x^3\left(\frac{1}{4}\ln x - \frac{1}{8}\right) + \frac{1}{4}x^3 (\ln x)^2$$

Note: One need not include the $\frac{1}{8}x^3$ term in the above, since x^3 is a homogeneous solution, but variation of parameters picks out one particular solution.

2d (10 **pts.**) Give a general solution to $(*)$.

$$y = y_h + y_p = C_1 x + C_2 x^3 - x^3 \left(\frac{1}{4} \ln x \right) + \frac{1}{4} x^3 (\ln x)^2$$