Exam III Solutions for Ma 221 2004 Fall.

1 Exam IIIA

1 (25 **pts.**) Use Laplace Transforms to solve

$$y'' + 2y' + 5y = 0 \quad y(0) = 1 \;\; y'(0) = -3$$

Solution: We take the Laplace transform of both sides and get

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2s\mathcal{L}\{y\} - 2y(0) + 5\mathcal{L}\{y\} = 0$$

Using the initial conditions we get

$$\left(s^2 + 2s + 5\right)\mathcal{L}\{y\} = s - 1$$

or

$$\mathcal{L}\{y\} = \frac{s-1}{s^2 + 2s + 5}$$

We must find

$$\mathcal{L}^{-1}\left\{\frac{s-1}{s^2 + 2s + 5}\right\}$$

$$\frac{s-1}{s^2 + 2s + 5} = \frac{s-1}{(s+1)^2 + 4} = \frac{s+1}{(s+1)^2 + 4} + \frac{-2}{(s+1)^2 + 4}$$

Therefore

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s-1}{s^2 + 2s + 5}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2 + 4}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2 + 4}\right\} \\ &= e^{-t}\cos 2t - e^{-t}\sin t\end{aligned}$$

2a (10 pts.) Use the definition of the Laplace transform to find $\mathcal{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 0 & 0 \le t < 4 \\ 2e^{-3t} & 4 \le t < \infty \end{cases}$$

Solution:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = 2\int_4^\infty e^{-(s+3)t} dt = -2 \lim_{R\to\infty} \left.\frac{e^{-(s+3)t}}{s+3}\right|_4^R = -\frac{2}{s+3} \lim_{R\to\infty} \left[e^{-(s+3)R} - e\right.$$

2b (15 **pts.**) Find $\mathcal{L}^{-1}\left\{\frac{(s+5)(s+3)}{s(s+2)(s+6)}\right\}$.

Solution:

$$\frac{(s+5)(s+3)}{s(s+2)(s+6)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+6}$$

Multiplying by s and setting $s = 0$ gives

$$A = \frac{5(3)}{2(6)} = \frac{5}{4}$$

Similarly multiplying by $(s+2)$ and setting $s = -2$ gives

$$B = \frac{3(1)}{-2(4)} = -\frac{3}{8}$$

and multiplying by $(s+6)$ and setting $s = -6$ gives

$$C = \frac{(-1)(-3)}{(-6)(-4)} = \frac{1}{8}$$

Thus

$$\frac{(s+5)(s+3)}{s(s+2)(s+6)} = \left(\frac{5}{4}\right)\frac{1}{s} - \left(\frac{3}{8}\right)\frac{1}{s+2} + \left(\frac{1}{8}\right)\frac{1}{s+6}$$

so

$$\mathcal{L}^{-1}\left\{\frac{(s+5)(s+3)}{s(s+2)(s+6)}\right\} = \frac{5}{4} - \left(\frac{3}{8}\right)e^{-2t} + \left(\frac{1}{8}\right)e^{-6t}$$

3 (25 **pts.**) Find the first <u>six</u> non-zero terms in the series solution near $x = 0$ of the equation

$$y'' - xy = 0$$

Give the recurrence relation also.
Solution:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=1}^{\infty} a_n (n) x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} \end{aligned}$$

Substituting into the DE gives

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

We shift the second summation by letting $k - 2 = n + 1$ or $n = k - 3$ and get, since $n = 0 \Rightarrow k = 3$

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} - \sum_{k=3}^{\infty} a_{k-3} x^{k-2} = 0$$

Replacing n and k by m yields

$$(2)(1) a_2 + \sum_{m=3}^{\infty} \{a_m (m)(m-1) - a_{m-3}\} x^{m-2} = 0$$

Thus $a_2 = 0$ and we have the recurrence relation

$$a_m = \frac{1}{m(m-1)} a_{m-3} \qquad m = 3, 4, \ldots$$

Thus

$$\begin{aligned}
a_3 &= \frac{1}{6}a_0 \\
a_4 &= \frac{1}{12}a_1 \\
a_5 &= 0 \\
a_6 &= \frac{1}{30}a_3 = \frac{1}{180}a_0 \\
a_7 &= \frac{1}{42}a_4 = \frac{1}{(42)(12)}a_1
\end{aligned}$$

Thus

$$\begin{aligned}
y(x) &= a_0 + a_1 x + a_2 x^2 + \cdots \\
&= a_0\left[1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots\right] + a_1\left[x + \frac{1}{12}x^4 + \frac{1}{(42)(12)}x^7 + \cdots\right]
\end{aligned}$$

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4 (25 **pts.**) Find the eigenvalues and eigenfunctions for

$$y'' + (\lambda + 4)\, y = 0 \qquad y(0) = \; y(1) = 0$$

Be sure to consider all values of λ.

Solution: The characteristic equation is

$$r^2 + (\lambda + 4) = 0$$

so

$$r = \pm\sqrt{-\lambda - 4}$$

There are 3 cases to consider:

I. $-\lambda - 4 > 0$, that is $\lambda < -4$. Let $\alpha^2 = -\lambda - 4, \alpha \neq 0$. Then $r = \pm\alpha$ and $y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$. The boundary conditions imply

$$\begin{aligned}
y(0) &= c_1 + c_2 = 0 \quad \text{or } c_1 = -c_2 \\
y(1) &= c_1 e^{\alpha} + c_2 e^{-\alpha} = 0 \text{ or } c_1\left(e^{\alpha} - e^{-\alpha}\right) = 0
\end{aligned}$$

Since $e^{\alpha} - e^{-\alpha} \neq 0$, we see that $c_1 = c_2 = 0$, so $y = 0$ and there are no eigenvalues for $\lambda < -4$.

II. $\lambda = -4$. In this case $r = 0$ is a repeated root so $y(x) = c_1 + c_2 x$. The boundary conditions imply $c_1 = c_2$, so $y = 0$ and $\lambda = -4$ is not an eigenvalue.

III. $-\lambda - 4 < 0$, that is $\lambda > -4$. Let $\beta^2 = -(-\lambda - 4) = \lambda + 4$. Then $r = \pm\beta i$ and $y(x) = c_1 \sin\beta x + c_2 \cos\beta x$. The boundary conditions imply

$$\begin{aligned} y(0) &= c_2 = 0 \\ y(1) &= c_1 \sin\beta = 0 \end{aligned}$$

Thus

$$\beta = n\pi, \quad n = 1, 2, 3, \ldots$$

so the eigenvalues are

$$\lambda_n = \beta^2 - 4 = n^2\pi^2 - 4, \quad n = 1, 2, 3, \ldots$$

and the eigenfunctions are

$$y_n(x) = A_n \sin(n\pi x)$$

2 Exam IIIB

1 (25 **pts.**) Use Laplace Transforms to solve

$$y'' - 2y' + 5y = 0 \qquad y(0) = 2 \;\; y'(0) = -5$$

Solution: We take the Laplace transform of both sides and get

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 2s\mathcal{L}\{y\} + 2y(0) + 5\mathcal{L}\{y\} = 0$$

Using the initial conditions we get

$$\left(s^2 + 2s + 5\right)\mathcal{L}\{y\} = 2s - 9$$

or

$$\mathcal{L}\{y\} = \frac{2s-9}{s^2 - 2s + 5}$$

We must find

$$\mathcal{L}^{-1}\left\{\frac{2s-9}{s^2 - 2s + 5}\right\}$$

$$\frac{2s-9}{s^2-2s+5} = \frac{2s-9}{(s-1)^2+4} = \frac{2(s-1)}{(s-1)^2+4} + \frac{-7}{(s-1)^2+4}$$

Therefore

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{2s-9}{s^2-2s+5}\right\} &= 2\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+2^2}\right\} - \frac{7}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\} \\ &= 2e^t\cos 2t - \frac{7}{2}e^t\sin 2t \end{aligned}$$

2a (10 pts.) Use the definition of the Laplace transform to find $\mathcal{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 0 & 0 \le t < 2 \\ 4e^{3t} & 2 \le t < \infty \end{cases}$$

Solution:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = 4\int_2^\infty e^{-(s-3)t}dt = -4\lim_{R\to\infty} \left.\frac{e^{-(s-3)t}}{s-3}\right|_2^R = -\frac{4}{s-3}\lim_{R\to\infty}\left[e^{-(s-3)R} - e\right.$$

2b (15 **pts.**) Find $\mathcal{L}^{-1}\left\{\frac{(s-5)(s-3)}{s(s-2)(s-6)}\right\}$.

Solution:

$$\frac{(s-5)(s-3)}{s(s-2)(s-6)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s-6}$$

or

$$(s-5)(s-3) = A(s-2)(s-6) + Bs(s-6) + Cs(s-2)$$

Setting $s=0$ gives

$$A = \frac{(-5)(-3)}{(-2)(-6)} = \frac{5}{4}$$

Similarly setting $s=2$ gives

$$B = \frac{(-3)(-1)}{(2)(-4)} = -\frac{3}{8}$$

and setting $s=6$ gives

$$C = \frac{(1)(3)}{(6)(4)} = \frac{1}{8}$$

Thus
$$\frac{(s-5)(s-3)}{s(s-2)(s-6)} = \left(\frac{5}{4}\right)\frac{1}{s} - \left(\frac{3}{8}\right)\frac{1}{s-2} + \left(\frac{1}{8}\right)\frac{1}{s-6}$$
so
$$\mathcal{L}^{-1}\left\{\frac{(s-5)(s-3)}{s(s-2)(s-6)}\right\} = \frac{5}{4} - \left(\frac{3}{8}\right)e^{2t} + \left(\frac{1}{8}\right)e^{6t}$$

3 (25 **pts.**) Find the first <u>six</u> non-zero terms in the series solution near $x = 0$ of the equation

$$y'' - x^2 y = 0$$

Also give the recurrence relation.
Solution:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=1}^{\infty} a_n (n) x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} \end{aligned}$$

Substituting into the DE gives

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

We shift the first summation by letting $k = n-2$ or $n = k+2$ and get, since $n = 2 \Rightarrow k = 0$

$$\sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) x^k - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

We shift the second summation by letting $k = n+2$ or $n = k-2$ and get, since $n = 0 \Rightarrow k = 2$

$$\sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) x^k - \sum_{k=2}^{\infty} a_{k-2} x^k = 0$$

We observe that the first series has two more terms than the second, pull them out and combine the rest to obtain

$$(2)\,(1)\,a_2 + (3)\,(2)\,a_3 x + \sum_{k=2}^{\infty} \{a_{k+2}\,(k+2)\,(k+1) - a_{k-2}\}\, x^k = 0$$

Thus $a_2 = 0$ and $a_3 = 0$ and we have the recurrence relation

$$a_{k+2} = \frac{1}{(k+2)\,(k+1)} a_{m-2} \qquad k = 2, 3, , \ldots$$

Thus

$$\begin{aligned}
a_4 &= \frac{1}{4 \cdot 3} a_0 \\
a_5 &= \frac{1}{5 \cdot 4} a_1 \\
a_6 &= a_7 = 0 \\
a_8 &= \frac{1}{8 \cdot 7} a_4 = \frac{1}{8 \cdot 7 \cdot 4 \cdot 3} a_0 \\
a_9 &= \frac{1}{9 \cdot 8} a_5 = \frac{1}{9 \cdot 8 \cdot 5 \cdot 4} a_1
\end{aligned}$$

Thus

$$\begin{aligned}
y\,(x) &= a_0 + a_1 x + a_2 x^2 + \cdots \\
&= a_0 \left[1 + \frac{1}{4 \cdot 3} x4 + \frac{1}{8 \cdot 7 \cdot 4 \cdot 3} x^8 + \cdots \right] + a_1 \left[x + \frac{1}{5 \cdot 4} x^5 + \frac{1}{9 \cdot 8 \cdot 5 \cdot 4} x^9 + \cdots \right]
\end{aligned}$$

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4 (25 **pts.**) Find the eigenvalues and eigenfunctions for

$$y'' + (\lambda + 4)\, y = 0 \qquad y'\,(0) = \; y'\,(1) = 0$$

Be sure to consider all values of λ.

Solution: The characteristic equation is

$$r^2 + (\lambda + 4) = 0$$

so

$$r = \pm\sqrt{-\lambda - 4}$$

There are 3 cases to consider:

I. Two real roots, corresponding to a positive discriminant. $-\lambda - 4 > 0$, that is $\lambda < -4$. Let $\alpha^2 = -\lambda - 4, \alpha \neq 0$. Then $r = \pm\mu$ and $y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$ *and* $y'(x) = c_1 \mu e^{\mu x} - c_2 \mu e^{-\mu x}$. The boundary conditions imply

$$\begin{aligned} y'(0) &= c_1 - c_2 = 0 \quad \text{or } c_1 = c_2 \\ y'(1) &= \mu\left(c_1 e^{\alpha} - c_2 e^{-\alpha}\right) = 0 \text{ or } c_1 \mu\left(e^{\alpha} - e^{-\alpha}\right) = 0 \end{aligned}$$

Since $e^{\alpha} - e^{-\alpha} \neq 0$, we see that $c_1 = c_2 = 0$, so $y = 0$ and there are no eigenvalues for $\lambda < -4$.

II. A single repeated real root correspnding to the discriminant having value zero. $\lambda = -4$. In this case $r = 0$ is a repeated root so $y(x) = c_1 + c_2 x$ *and* $y'(x) = c_2$. The boundary conditions imply $c_2 = 0$, so $y = c$ is a solution and $\lambda = -4$ is an eigenvalue.

III.Complex roots corresponding to a negative discriminant. $-\lambda - 4 < 0$, that is $\lambda > -4$. Let $\mu^2 = -(-\lambda - 4) = \lambda + 4$. Then $r = \pm\mu i$ and $y(x) = c_1 \cos\mu x + c_2 \sin\mu x$ *and* $y'(x) = -c_1 \mu \sin\mu x + c_2 \cos\mu x$. The boundary conditions imply

$$\begin{aligned} y'(0) &= c_2 = 0 \\ y(1) &= c_1 \mu \sin\mu = 0 \end{aligned}$$

Thus

$$\mu_n = n\pi, \quad n = 1, 2, 3, \ldots$$

so the eigenvalues are

$$\lambda_n = \mu_n^2 - 4 = n^2\pi^2 - 4, \quad n = 1, 2, 3, \ldots$$

and the eigenfunctions are

$$y_n(x) = A_n \cos(n\pi x), \quad n = 1, 2, 3, ...$$