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IDN:_________________________________ Recitation Section ____

Ma 221 **Exam II A Solutions** **05F**

I pledge my honor that I have abided by the Stevens Honor System.

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You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1a ________

#1b ________

#1c ________

#2a ________

#2b ________

#2c ________

#2d ________

3 _________

Total Score ________

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1. Consider the differential equation

$$y'' + 4y = 3e^{2x} + \sin 2x + 8x^2$$

1 a (7 **pts**.) Find the homogeneous solution of this equation.

Solution: The characteristic equation is $p(r) = r^2 + 4 = 0,$ so $r = \pm 2i.$ Thus

$$y_h = c_1 \sin 2x + c_2 \cos 2x$$

1 b (25 **pts**.) Find a particular solution of this equation.

Solution: We first find a particular solution for $3e^{2x}$. Since $p(2) = 8 \neq 0,$ then

$$y_{p_1} = \frac{3e^{2x}}{8}$$

To find a particular solution for $\sin 2x$, there are two approaches:

Approach 1: (using complex variables)

$$y'' + 4y = \sin 2x$$

and a companion equation

$$v'' + 4v = \cos 2x$$

We multiply the first equation by i and add it to the second to get

$$w'' + 4w = e^{2ix}$$

where $w = iy + v.$ Then since $p(2i) = 0$ and $p'(2i) = 4i \neq 0$

$$w_{p_2} = \frac{xe^{2ix}}{4i} = -\frac{ix}{4}(\cos 2x + i \sin 2x)$$

Thus

$$y_{p_2} = imag w_{p_2} = -\frac{x \cos 2x}{4}.$$

Approach 2: (without complex variables)

$$y'' + 4y = \sin 2x$$

Since $\sin 2x$ is a solution of the homogeneous equation we let

$$y_{p_2} = x(A \sin 2x + B \cos 2x)$$

Then, $y'_{p_2} = \frac{d}{dx}\{x(A \sin 2x + B \cos 2x)\} = B \cos 2x + A \sin 2x + x(2A \cos 2x - 2B \sin 2x)$

and $y''_{p_2} = \frac{d}{dx}\{B \cos 2x + A \sin 2x + x(2A \cos 2x - 2B \sin 2x)\} =$

$4A \cos 2x - 4B \sin 2x + x(-4B \cos 2x - 4A \sin 2x)$

Substituting into the DE:

$4A \cos 2x - 4B \sin 2x + x(-4B \cos 2x - 4A \sin 2x) + 4x(A \sin 2x + B \cos 2x) = \sin 2x$

$\Rightarrow 4A \cos 2x - 4B \sin 2x = \sin 2x$

So, $4A = 0 \Rightarrow A = 0$

$-4B = 1 \Rightarrow B = -\frac{1}{4}$

So,

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$$y_{p_2} = -\frac{1}{4}x\cos 2x$$

To find a particular solution for $8x^2$ we let

$$y_{p_3} = a_0 + a_1x + a_2x^2$$

Then $y'_{p_3} = a_1 + 2a_2x$ and $y''_{p_3} = 2a_2$ so the DE implies

$$2a_2 + 4a_0 + 4a_1x + 4a_2x^2 = 8x^2$$

Then $a_2 = 2, a_1 = 0$ and $a_0 = -\frac{1}{2}a_2 = -1,$ so

$$y_{p_3} = -1 + 2x^2$$

Therefore

$$y_p = y_{p_1} + y_{p_2} + y_{p_3} = \frac{3e^{2x}}{8} - \frac{x\cos 2x}{4} - 1 + 2x^2$$

1 c (5 **pts**.) Give a general solution of this equation.

$$y = y_h + y_p = c_1\sin 2x + c_2\cos 2x + \frac{3e^{2x}}{8} - \frac{x\cos 2x}{4} - 1 + 2x^2$$

2 Consider the differential equation

$$y'' + 2y' + y = t^5e^{-t} \qquad (*)$$

2 a (**8 pts**.) Find two linearly independent solutions of the homogeneous equation corresponding to $(*)$ and give the homogeneous solution.

Solution: The characteristic polynomial is $p(r) = r^2 + 2r + 1 = (r+1)^2$ so $r = -1$ is a repeated root and e^{-t} and te^{-t} are LI solutions.

$$y_h = c_1e^{-t} + c_2te^{-t}$$

2 b (**10 pts**.) Find the value of the Wronskian of the two linearly independent solutions you found in 2a.

$$W[e^{-t}, te^{-t}] = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} = e^{-2t}$$

2c (25 **pts**.) Use Variation of Parameters to find a particular solution to $(*)$.

$$y_p = v_1e^{-t} + v_2te^{-t}$$

The two equations for v'_1 and v'_2 are

$$v'_1e^{-t} + v'_2te^{-t} = 0$$
$$-v'_1e^{-t} + v'_2(e^{-t} - te^{-t}) = t^5e^{-t}$$

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$$v_1' = \frac{\begin{vmatrix} 0 & te^{-t} \\ t^5e^{-t} & e^{-t} - te^{-t} \end{vmatrix}}{e^{-2t}} = -t^6$$

$$v_2' = \frac{\begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & t^5e^{-t} \end{vmatrix}}{e^{-2t}} = t^5$$

Thus

$$v_1 = -\frac{t^7}{7}$$
$$v_2 = \frac{t^6}{6}$$

or

$$v_1 = -\int \frac{y_2 f(t)}{aW[y_1, y_2]}dt = -\int \frac{te^{-t}(t^5e^{-t})}{1\left(e^{-2t}\right)}dt = -\int t^6 dt = -\frac{t^7}{7}$$

$$v_2 = \int \frac{y_1 f(t)}{aW[y_1, y_2]}dt = \int \frac{e^{-t}(t^5e^{-t})}{1\left(e^{-2t}\right)}dt = \int t^5 dt = \frac{t^6}{6}$$

so

$$y_p = -\frac{t^7}{7}e^{-t} + \frac{t^7}{6}e^{-t} = \frac{1}{42}t^7e^{-t}$$

2 d (**5 pts**.) Give a general solution to $(*)$.

$$y = y_h + y_p = c_1e^{-t} + c_2te^{-t} + \frac{1}{42}t^7e^{-t}$$

$y'' + 2y' + y = t^5e^{-t}$, Exact solution is: $y(t) = \frac{1}{42}t^7e^{-t} + C_1e^{-t} + C_2te^{-t}$

3 (**15 pts**.) Solve the equation

$$x^2y'' + 2xy' + y = 0$$

Solution: Since $p = 2$ and $q = 1$, the indicial equation is

$$r^2 + (2-1)r + 1 = r^2 + r + 1 = 0$$

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

We get complex conjugate roots with the real part $\alpha = -\frac{1}{2}$ and the imaginary part $\beta = \frac{\sqrt{3}}{2}$.
Thus

$$y = c_1x^{-\frac{1}{2}}\sin\left(\frac{\sqrt{3}}{2}\ln x\right) + c_2x^{-\frac{1}{2}}\cos\left(\frac{\sqrt{3}}{2}\ln x\right)$$