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IDN:_________________________________ Recitation Section ____

Ma 221 **Exam II B Solutions** **05F**

I pledge my honor that I have abided by the Stevens Honor System.

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You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1a ________

#1b ________

#1c ________

#2a ________

#2b ________

#2c ________

#2d ________

3 _________

Total Score ________

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1. Consider the differential equation

$$y'' + 9y = 4e^{3x} + \cos 3x - 9x^2$$

1 a (7 **pts**.) Find the homogeneous solution of this equation.

Solution: The characteristic equation is $p(r) = r^2 + 9 = 0,$ so $r = \pm 3i.$ Thus

$$y_h = c_1 \sin 3x + c_2 \cos 3x$$

1 b (25 **pts**.) Find a particular solution of this equation.

Solution: We first find a particular solution for $4e^{3x}$. Since $p(3) = 18 \neq 0,$ then

$$y_{p_1} = \frac{4e^{3x}}{18} = \frac{2e^{3x}}{9}$$

To find a particular solution for $\cos 3x$, there are two approaches:

Approach 1: (using complex variables)

$$y'' + 9y = \cos 3x$$

and a companion equation

$$v'' + 9v = \sin 3x$$

We multiply the first equation by i and add it to the second to get

$$w'' + 9w = e^{3ix}$$

where $w = y + iv.$ Then since $p(3i) = 0$ and $p'(3i) = 6i \neq 0$

$$w_{p_2} = \frac{xe^{3ix}}{6i} = -\frac{ix}{6}(\cos 3x + i \sin 3x)$$

Thus

$$y_{p_2} = \text{real}\, w_{p_2} = \frac{x \sin 3x}{6}.$$

Approach 2: (without complex variables)

$$y'' + 9y = \cos 3x$$

Since $\cos 3x$ is a solution of the homogeneous equation we let

$$y_{p_2} = x(A \cos 3x + B \sin 3x)$$

Then, $y'_{p_2} = \frac{d}{dx}\{x(A\cos 3x + B \sin 3x)\} = A \cos 3x + B \sin 3x + x(3B \cos 3x - 3A \sin 3x)$

and $y''_{p_2} = \frac{d}{dx}\{A \cos 3x + B \sin 3x + x(3B \cos 3x - 3A \sin 3x)\} =$

$6B \cos 3x - 6A \sin 3x + x(-9A \cos 3x - 9B \sin 3x)$

Substituting into the DE:

$6B \cos 3x - 6A \sin 3x + x(-9A \cos 3x - 9B \sin 3x) + 9x(A \cos 3x + B \sin 3x) = \cos 3x$

$\Rightarrow 6B \cos 2x - 6A \sin 2x = \cos 3x$

So, $-6A = 0 \Rightarrow A = 0$

$6B = 1 \Rightarrow B = \frac{1}{6}$

So,

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$$y_{p_2} = \frac{1}{6} x \sin 3x$$

To find a particular solution for $-9x^2$ we let

$$y_{p_3} = a_0 + a_1 x + a_2 x^2$$

Then $y'_{p_3} = a_1 + 2a_2 x$ and $y''_{p_3} = 2a_2$ so the DE implies

$$2a_2 + 9a_0 + 9a_1 x + 9a_2 x^2 = -9x^2$$

Then $a_2 = -1, a_1 = 0$ and $a_0 = -\frac{2}{9}a_2 = \frac{2}{9}$, so

$$y_{p_3} = \frac{2}{9} - x^2$$

Therefore

$$y_p = y_{p_1} + y_{p_2} + y_{p_3} = \frac{2e^{3x}}{9} + \frac{x \sin 3x}{6} + \frac{2}{9} - x^2$$

1 c (5 **pts**.) Give a general solution of this equation.

$$y = y_h + y_p = c_1 \sin 3x + c_2 \cos 3x + \frac{2e^{3x}}{9} + \frac{x \sin 3x}{6} + \frac{2}{9} - x^2$$

2 Consider the differential equation

$$y'' - 2y' + y = t^6 e^t \qquad (*)$$

2 a (**8 pts**.) Find two linearly independent solutions of the homogeneous equation corresponding to $(*)$ and give the homogeneous solution.

Solution: The characteristic polynomial is $p(r) = r^2 - 2r + 1 = (r-1)^2$ so $r = 1$ is a repeated root and e^t and te^t are LI solutions.

$$y_h = c_1 e^t + c_2 t e^t$$

2 b (**10 pts**.) Find the value of the Wronskian of the two linearly independent solutions you found in 2a.

$$W[e^{-t}, te^{-t}] = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t}$$

2c (25 **pts**.) Use Variation of Parameters to find a particular solution to $(*)$.

$$y_p = v_1 e^t + v_2 t e^t$$

The two equations for v'_1 and v'_2 are

$$v'_1 e^t + v'_2 t e^t = 0$$
$$v'_1 e^t + v'_2 (e^t + te^t) = t^6 e^t$$

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$$v_1' = \frac{\begin{vmatrix} 0 & te^t \\ t^6 e^t & e^t + te^t \end{vmatrix}}{e^{2t}} = -t^7$$

$$v_2' = \frac{\begin{vmatrix} e^t & 0 \\ e^t & t^6 e^t \end{vmatrix}}{e^{2t}} = t^6$$

Thus

$$v_1 = -\frac{t^8}{8}$$
$$v_2 = \frac{t^7}{7}$$

or

$$v_1 = -\int \frac{y_2 f(t)}{aW[y_1, y_2]} dt = -\int \frac{te^t\left(t^6 e^t\right)}{1\left(e^{2t}\right)} dt = -\int t^7 dt = -\frac{t^8}{8}$$

$$v_2 = \int \frac{y_1 f(t)}{aW[y_1, y_2]} dt = \int \frac{e^t\left(t^6 e^t\right)}{1\left(e^{2t}\right)} dt = \int t^6 dt = \frac{t^7}{7}$$

so

$$y_p = -\frac{t^8}{8} e^{-t} + \frac{t^8}{7} e^t = \frac{1}{56} t^8 e^t$$

2 d (**5 pts**.) Give a general solution to $(*)$.

$$y = y_h + y_p = c_1 e^t + c_2 te^t + \frac{1}{56} t^8 e^t$$

$y'' - 2y' + y = t^6 e^t$, Exact solution is: $C_1 e^t + C_2 te^t + \frac{1}{56} t^8 e^t$

3 (**15 pts**.) Solve the equation

$$x^2 y'' + 2xy' + 3y = 0$$

Solution: Since $p = 2$ and $q = 3$, the indicial equation is

$$r^2 + (2-1)r + 3 = r^2 + r + 3 = 0$$

$$r = \frac{-1 \pm \sqrt{1-12}}{2} = \frac{-1 \pm i\sqrt{11}}{2}$$

We get complex conjugate roots with the real part $\alpha = -\frac{1}{2}$ and the imaginary part $\beta = \frac{\sqrt{11}}{2}$.
Thus

$$y = c_1 x^{-\frac{1}{2}} \sin\left(\frac{\sqrt{11}}{2} \ln x\right) + c_2 x^{-\frac{1}{2}} \cos\left(\frac{\sqrt{11}}{2} \ln x\right)$$