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Ma 221 Exam IIIA Solutions 05F

I pledge my honor that I have abided by the Stevens Honor System.
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You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1 ________

#2 ________

#3 ________

#4 ________

Total Score ________

Note: A table of Laplace Transforms is given at the end of the exam.

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1 (25 **pts**.) Use Laplace Transforms to solve

$$y'' - y = e^{2t} \quad y(0) = 0 \;\; y'(0) = 1$$

Solution: Taking the Laplace transform of both sides of the DE yields

$$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = \frac{1}{s-2}$$

or

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - \mathcal{L}\{y\} = \frac{1}{s-2}$$

Thus

$$(s^2-1)\mathcal{L}\{y\} = \frac{1}{s-2} + 1 = \frac{s-1}{s-2}$$

so

$$\mathcal{L}\{y\} = \frac{s-1}{(s-2)(s^2-1)} = \frac{1}{(s+1)(s-2)}$$

Now

$$\frac{1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$$

and $A = -\frac{1}{3}$ and $B = \frac{1}{3}$ so

$$\mathcal{L}\{y\} = -\left(\frac{1}{3}\right)\left(\frac{1}{s+1} - \frac{1}{s-2}\right)$$

Thus

$$y(t) = -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t}$$

2a (**10 pts**.) Use the definition of the Laplace transform to find $\mathcal{L}\{t\}$. Assume $s > 0$.

Solution:

$$\mathcal{L}\{t\} = \int_0^\infty te^{-st}dt = \lim_{R\to\infty}\int_0^R te^{-st}dt$$

Integrating by parts with $u = t$ and $dv = e^{-st}$ we have $du = dt$ and $v = -\frac{1}{s}e^{-st}$ so

$$\mathcal{L}\{t\} = \lim_{R\to\infty}\left[-\frac{1}{s}te^{-st}\Big]_0^R - \int_0^R\left(-\frac{1}{s}e^{-st}\right)dt\right] = \lim_{R\to\infty}\left[-\frac{1}{s}R\,e^{-sR} - \frac{1}{s^2}(e^{-st})_0^R\right] = \lim_{R\to\infty}\left[-\frac{1}{s}R\,e^{-sR} - \frac{1}{s^2}(e^{-sR}) + \frac{1}{s^2}\right] = \frac{1}{s^2}$$

2b (15 **pts**.) Find $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4s+9}\right\}$.

Solution:

$$\frac{s+1}{s^2+4s+9} = \frac{s+1}{s^2+4s+4+5} = \frac{s+1}{(s+2)^2+5} = \frac{s+2-1}{(s+2)^2+5} = \frac{s+2}{(s+2)^2+\left(\sqrt{5}\right)^2} - \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{(s+2)^2+\left(\sqrt{5}\right)^2}$$

so

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4s+13}\right\} = e^{-2t}\cos\left(\sqrt{5}\,t\right) - \frac{1}{\sqrt{5}}e^{-2t}\sin\left(\sqrt{5}\,t\right)$$

3 (25 **pts**.) Find the first <u>six</u> non-zero terms in the series solution near $x = 0$ of the equation

$$y'' - xy' + 2y = 0$$

Give the recurrence relation also.

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Solution:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n (n) x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

Substituting into the DE we have

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} - \sum_{n=1}^{\infty} a_n (n) x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

We shift the first sum in the above equation by letting $k = n-2$ or $n = k+2$ and combine the second and third sums. This yields

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k + \sum_{n=1}^{\infty} a_n (2-n) x^n + 2a_0 = 0$$

Replacing k and n by m we have

$$\sum_{m=1}^{\infty} \{a_{m+2}(m+2)(m+1) + a_m(2-m)\} x^m + 2(1)a_2 + 2a_0 = 0$$

Thus $a_2 = -a_0$ and

$$a_{m+2}(m+2)(m+1) + a_m(2-m) = 0$$

or

$$a_{m+2} = \frac{m-2}{(m+2)(m+1)} a_m \text{ for } m = 1,2,3,\ldots$$

Thus

$$a_3 = \frac{1}{3(2)} a_1 = \frac{1}{6} a_1$$

$$a_4 = 0$$

$$a_5 = \frac{1}{5(4)} a_3 = \frac{1}{5(4)(3)(2)} a_1 = \frac{1}{120} a_1$$

$$a_6 = 0$$

$$a_7 = \frac{3}{7(6)} a_5 = \frac{1}{7(6)(5)(4)(2)} a_1 = \frac{1}{1680} a_1$$

Hence

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= a_0(1-x^2) + a_1\left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{1680}x^7 + \cdots\right)$$

SNB check: $y'' - xy' + 2y = 0$, Series solution is:
$\{y(0) + xy'(0) - x^2 y(0) - \frac{1}{6}x^3 y'(0) - \frac{1}{120}x^5 y'(0) - \frac{1}{1680}x^7 y'(0) + O(x^9)\}$

4 (25 **pts**.) Find the eigenvalues and eigenfunctions for

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$$y'' + \lambda y = 0 \quad y(0) = y'(\pi) = 0$$

Be sure to consider all values of λ.

Solution: There are three cases to consider.

I. $\lambda < 0$. Let $\lambda = -\alpha^2$. Then we have

$$y'' - \alpha^2 y = 0$$

so

$$y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

$$y(0) = c_1 + c_2 = 0$$

so $c_1 = -c_2$ and $y(x) = c_1(e^{\alpha x} - e^{-\alpha x})$. Hence $y'(x) = c_1(\alpha)(e^{\alpha x} + e^{-\alpha x})$.

$$y'(\pi) = c_1(\alpha)(e^{\alpha\pi} + e^{-\alpha\pi}) = 0$$

Thus $c_1 = c_2 = 0$, and $y = 0$ and there are no eigenvalues for $\lambda < 0$.

II. $\lambda = 0$. The $y(x) = c_1 x + c_2$. $y(0) = 0$ implies that $c_2 = 0$, whereas $y'(\pi) = c_1 = 0$. Hence no eigenvalues for $\lambda = 0$.

III. $\lambda > 0$. Let $\lambda = \beta^2$, where $\beta \neq 0$. Then we have

$$y'' + \beta^2 y = 0$$

and $y(x) = c_1 \cos\beta x + c_2 \sin\beta x$. $y(0) = c_1 = 0$. Thus $y(x) = c_2 \sin\beta x$ so $y'(x) = c_2 \beta \cos\beta x$. $y'(\pi) = 0$ implies that

$$\cos\beta\pi = 0$$

so

$$\beta\pi = (2n+1)\frac{\pi}{2} \quad n = 0, 1, 2, \ldots$$

Thus $\beta = \frac{2n+1}{2}$ and the eigenvalues are

$$\lambda = \beta^2 = \left(\frac{2n+1}{2}\right)^2 \quad n = 0, 1, 2, \ldots$$

with corresponding eigenfunctions

$$y_n(x) = a_n \sin\left(\frac{2n+1}{2}\right)x \quad n = 0, 1, 2, \ldots$$

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Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$		
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	$n \geq 1$	$s > 0$
e^{at}	$\frac{1}{s-a}$		$s > a$
$\sin bt$	$\frac{b}{s^2 + b^2}$		$s > a$
$\cos bt$	$\frac{s}{s^2 + b^2}$		$s > a$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		

$$\int u\,dv = uv - \int v\,du$$