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IDN:_________________________________ Recitation Section ____

Ma 221 **Exam II A Solutions** **07S**

I pledge my honor that I have abided by the Stevens Honor System.

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You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1a ________

#1b ________

#1c ________

#2a ________

#2b ________

#2c ________

#2d ________

#3 _________

Total Score ________

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1. Consider the differential equation

$$y'' + 9y = -26e^{-2x} + 27x^2 + 2\sin 3x$$

1 a (7 **pts**.) Find the homogeneous solution of this equation.

Solution: The characteristic equation is $p(\lambda) = \lambda^2 + 9 = 0$ so $\lambda = \pm 3i$. Thus

$$y_h = c_1 \sin 3x + c_2 \cos 3x$$

1 b (25 **pts**.) Find a particular solution of this equation.

Solution: We find a particular solution for each function on the right hand side.

y_{p_1} for $-26e^{-2x}$. Since $p(-2) = 13$, then

$$y_{p_1} = \frac{-26e^{-2x}}{13} = -2e^{-2x}$$

y_{p_2} for $27x^2$

$$y_{p_2} = A_0 + A_1 x + A_2 x^2$$
$$y'_{p_2} = A_1 + 2A_2 x$$
$$y''_{p_2} = 2A_2$$

Plugging into the DE yields

$$2A_2 + 9A_0 + 9A_1 x + 9A_2 x^2 = 27x^2$$

so $A_2 = 3$, $A_1 = 0$, and $2A_2 + 9A_0 = 6 + 9A_0 = 0$, so $A_0 = -\frac{2}{3}$. Thus

$$y_{p_2} = -\frac{2}{3} + 3x^2$$

y_{p_3} for $2\sin 3x$. Consider the equations

$$y'' + 9y = 2\sin 3x$$
$$v'' + 9v = 2\cos 3x$$

Letting $w = iy + v$ and multiplying the first equation by i and the adding it to the second equation yields

$$w'' + 9w = 2e^{3ix}$$

Since $3i$ is a root of the characteristic equation and $p(3i) = 6i \neq 0$,

$$w_{p_3} = \frac{2xe^{3ix}}{6i} = -\frac{1}{3} xi(\cos 3x + i\sin 3x)$$

Since

$$y_{p_3} = \operatorname{Im} w_{p_3} = -\frac{1}{3} x\cos 3x$$

Thus

$$y_p = y_{p_1} + y_{p_2} + +y_{p_3} = -2e^{-2x} - \frac{2}{3} + 3x^2 - \frac{1}{3} x\cos 3x$$

1 c (5 **pts**.) Give a general solution of this equation.

$$y = y_h + y_p = c_1 \sin 3x + c_2 \cos 3x - 2e^{-2x} - \frac{2}{3} + 3x^2 - \frac{1}{3} x\cos 3x$$

SNB check: $y'' + 9y = -26e^{-2x} + 27x^2 + 2\sin 3x$, Exact solution is:

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$\left\{C_{58}\cos 3x - C_{59}\sin 3x - \frac{1}{18e^{2x}}\left(12e^{2x} - 54x^2e^{2x} - e^{2x}\sin 3x + 6xe^{2x}\cos 3x + 36\right)\right\}$

2 Consider the differential equation

$$x^2y'' - 2xy' + 2y = x^2 \quad x > 0 \tag{$*$}$$

2 a (**8 pts**.) Find two linearly independent solutions of the homogeneous equation corresponding to $(*)$ and give the homogeneous solution.

Solution: This is an Euler equation with $p = -2$ and $q = 2$. The indicial equation is

$$r^2 + (p-1)r + q = r^2 - 3r + 2 = (r-2)(r-1) = 0$$

Thus $r = 1, 2$ and

$$y_h = c_1x + c_2x^2$$

2 b (**10 pts**.) Find the Wronskian of the two linearly independent solutions you found in 2a.

Solution:

$$W\left[x, x^2\right] = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2$$

2c (25 **pts**.) Find a particular solution to $(*)$.

Solution: $y_p = v_1x + v_2x^2$

$$v_1'x + v_2'x^2 = 0$$
$$v_1' + 2v_2'x = \frac{f}{a} = \frac{x^2}{x^2} = 1$$

Therefore

$$v_1' = \frac{\begin{vmatrix} 0 & x^2 \\ 1 & 2x \end{vmatrix}}{x^2} = -1$$

$$v_2' = \frac{\begin{vmatrix} x & 0 \\ 1 & 1 \end{vmatrix}}{x^2} = \frac{1}{x}$$

Therefore

$$v_1 = -x$$
$$v_2 = \ln x$$

Thus

$$y_p = -x^2 + x^2\ln x$$

There is no need to include the $-x^2$ term, since x^2 is a homogeneous solution. However, including it is not wrong.

2 d (**5 pts**.) Give a general solution to $(*)$.

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Solution:

$$y = y_h + y_p = c_1 x + c_2 x^2 + x^2 \ln x$$

SNB check: $x^2 y'' - 2xy' + 2y = x^2$, Exact solution is: $\left\{C_{86}x^2 + x^2\ln x + C_{85}x - x^2\right\}$

3 (15 pts.) Solve the initial value problem

$$y'' - 2y' + 5y = 0 \quad y(0) = 1 \;\; y'(0) = 0$$

Solution: $p(\lambda) = \lambda^2 - 2\lambda + 5 = 0$ so

$$\lambda = \frac{2 \pm \sqrt{4 - 4(5)}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

Thus

$$y = c_1 e^x \sin 2x + c_2 e^x \cos 2x$$

The initial conditions imply

$$y(0) = c_2 = 1$$

so

$$y(x) = c_1 e^x \sin 2x + e^x \cos 2x$$

Therefore

$$y'(x) = c_1 e^x \sin 2x + 2c_1 e^x \cos 2x + e^x \cos 2x - 2e^x \sin 2x$$

so

$$y'(0) = 2c_1 + 1 = 0$$

and $c_1 = -\frac{1}{2}$. Therefore

$$y(x) = -\frac{1}{2} e^x \sin 2x + e^x \cos 2x$$

SNB check:

$$\begin{aligned} y'' - 2y' + 5y &= 0 \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

, Exact solution is: $\left\{e^x \cos 2x - \frac{1}{2} e^x \sin 2x\right\}$