

Name: _____

Lecture Section ____

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Ma 221

Exam II B Solutions

07S

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You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1a _____

#1b _____

#1c _____

#2a _____

#2b _____

#2c _____

#2d _____

#3 _____

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Name: _____

Lecture Section ____

IDN: _____

Recitation Section ____

1. Consider the differential equation

$$y'' + 9y = -6e^{-3x} + 9x^2 + \sin 3x$$

1 a (7 pts.) Find the homogeneous solution of this equation.Solution: The characteristic equation is $p(\lambda) = \lambda^2 + 9 = 0$ so $\lambda = \pm 3i$. Thus

$$y_h = c_1 \sin 3x + c_2 \cos 3x$$

1 b (25 pts.) Find a particular solution of this equation.

Solution: We find a particular solution for each function on the right hand side.

 y_{p1} for $-6e^{-3x}$. Since $p(-3) = 18$, then

$$y_{p1} = \frac{-6e^{-3x}}{18} = -\frac{1}{3}e^{-3x}$$

 y_{p2} for $9x^2$

$$y_{p2} = A_0 + A_1x + A_2x^2$$

$$y'_{p2} = A_1 + 2A_2x$$

$$y''_{p2} = 2A_2$$

Plugging into the DE yields

$$2A_2 + 9A_0 + 9A_1x + 9A_2x^2 = 9x^2$$

so $A_2 = 1$, $A_1 = 0$, and $2A_2 + 9A_0 = 2 + 9A_0 = 0$, so $A_0 = -\frac{2}{9}$. Thus

$$y_{p2} = -\frac{2}{9} + x^2$$

 y_{p3} for $2 \sin 3x$. Consider the equations

$$y'' + 9y = \sin 3x$$

$$v'' + 9v = \cos 3x$$

Letting $w = iy + v$ and multiplying the first equation by i and then adding it to the second equation yields

$$w'' + 9w = e^{3ix}$$

Since $3i$ is a root of the characteristic equation and $p'(3i) = 6i \neq 0$,

$$w_{p3} = \frac{xe^{3ix}}{6i} = -\frac{1}{6}xi(\cos 3x + i \sin 3x)$$

Since

$$y_{p3} = \operatorname{Im} w_{p3} = -\frac{1}{6}x \cos 3x$$

Thus

$$y_p = y_{p1} + y_{p2} + y_{p3} = -\frac{1}{3}e^{-2x} - \frac{2}{9} + x^2 - \frac{1}{6}x \cos 3x$$

1 c (5 pts.) Give a general solution of this equation.

$$y = y_h + y_p = c_1 \sin 3x + c_2 \cos 3x - \frac{1}{3}e^{-2x} - \frac{2}{9} + x^2 - \frac{1}{6}x \cos 3x$$

SNB check: $y'' + 9y = -6e^{-2x} + 9x^2 + \sin 3x$, Exact solution is:

Name: _____

Lecture Section ____

IDN: _____

Recitation Section ____

$$\left\{ C_{31} \cos 3x - C_{32} \sin 3x - \frac{1}{36e^{3x}} (8e^{3x} - 36x^2 e^{3x} - e^{3x} \sin 3x + 6xe^{3x} \cos 3x + 12) \right\}$$

2 Consider the differential equation

$$x^2 y'' - 3xy' + 3y = x^2 \quad x > 0 \quad (*)$$

2 a (8 pts.) Find two linearly independent solutions of the homogeneous equation corresponding to (*) and give the homogeneous solution.Solution: This is an Euler equation with $p = -3$ and $q = 3$. The indicial equation is

$$r^2 + (p-1)r + q = r^2 - 4r + 3 = (r-3)(r-1) = 0$$

Thus $r = 1, 3$ and

$$y_h = c_1 x + c_2 x^3$$

2 b (10 pts.) Find the Wronskian of the two linearly independent solutions you found in 2a.

Solution:

$$W[x, x^3] = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3$$

2c (25 pts.) Find a particular solution to (*).Solution: $y_p = v_1 x + v_2 x^2$

$$v_1' x + v_2' x^3 = 0$$

$$v_1' + 3v_2' x^2 = \frac{f}{a} = \frac{x^2}{x^2} = 1$$

Therefore

$$v_1' = \frac{\begin{vmatrix} 0 & x^3 \\ 1 & 3x^2 \end{vmatrix}}{2x^3} = -\frac{1}{2}$$

$$v_2' = \frac{\begin{vmatrix} x & 0 \\ 1 & 1 \end{vmatrix}}{2x^3} = \frac{1}{2x^2}$$

Therefore

$$v_1 = -\frac{1}{2}x$$

$$v_2 = \frac{-1}{2x}$$

Thus

$$y_p = -\frac{1}{2}x^2 - \frac{1}{2}x^2 = -x^2$$

2 d (5 pts.) Give a general solution to (*).

Name: _____

Lecture Section ____

IDN: _____

Recitation Section ____

Solution:

$$y = y_h + y_p = c_1x + c_2x^3 - x^2$$

SNB check: $x^2y'' - 3xy' + 3y = x^2$, Exact solution is: $\{C_{59}x^3 - x^2 + C_{58}x\}$ **3 (15 pts.)** Solve the initial value problem

$$y'' - 2y' + 10y = 0 \quad y(0) = 1 \quad y'(0) = 0$$

Solution: $p(\lambda) = \lambda^2 - 2\lambda + 10 = 0$ so

$$\lambda = \frac{2 \pm \sqrt{4 - 4(10)}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

Thus

$$y = c_1e^x \sin 3x + c_2e^x \cos 3x$$

The initial conditions imply

$$y(0) = c_2 = 1$$

so

$$y(x) = c_1e^x \sin 3x + e^x \cos 3x$$

Therefore

$$y'(x) = c_1e^x \sin 3x + 3c_1e^x \cos 3x + e^x \cos 3x - 3e^x \sin 3x$$

so

$$y'(0) = 3c_1 + 1 = 0$$

and $c_1 = -\frac{1}{3}$. Therefore

$$y(x) = -\frac{1}{3}e^x \sin 3x + e^x \cos 3x$$

SNB check:

$$y'' - 2y' + 10y = 0$$

$$y(0) = 1 \quad , \text{ Exact solution is: } \left\{ e^x \cos 3x - \frac{1}{3} e^x \sin 3x \right\}$$

$$y'(0) = 0$$