

Name: _____

Lecture Section ____

Ma 221

**Exam IIIA Solutions
07S**

I pledge my honor that I have abided by the Stevens Honor System.

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1 _____

#2 _____

#3 _____

#4 _____

Total Score _____

Note: A table of Laplace Transforms is given at the end of the exam.

1 (25 pts.) Use Laplace Transforms to solve

$$y'' + 4y = 0 \quad y(0) = 2 \quad y'(0) = 2$$

Solutions:

$$s^2 \mathcal{L}\{y\} - sy(0) - y'(0) + 4\mathcal{L}\{y\} = 0$$

Thus

$$s^2 \mathcal{L}\{y\} - 2s - 2 + 4\mathcal{L}\{y\} = 0$$

So

$$\mathcal{L}\{y\} = \frac{2s+2}{s^2+4}$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left(\frac{2s+2}{s^2+4}\right) = 2\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + \mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) \\ &= 2\cos 2t + \sin 2t \end{aligned}$$

2a (10 pts.) $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\}$.

Solution:

Method 1, without complex variables:

$$\frac{1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

Multiplying by $s+1$ and setting $s = -1$, yields

$$\frac{1}{2} = A$$

Thus

$$\frac{1}{(s+1)(s^2+1)} = \frac{\frac{1}{2}}{s+1} + \frac{Bs+C}{s^2+1}$$

Setting $s = 0$ yields

$$\frac{1}{1} = \frac{1}{2} + \frac{C}{1}$$

so

$$C = \frac{1}{2}$$

and we have

$$\frac{1}{(s+1)(s^2+1)} = \frac{\frac{1}{2}}{s+1} + \frac{Bs + \frac{1}{2}}{s^2+1}$$

Setting $s = 1$, yields

$$\frac{1}{2(2)} = \frac{1}{4} + \frac{B + \frac{1}{2}}{2}$$

Thus

$$\frac{B}{2} + \frac{1}{4} = 0$$

so

$$B = -\frac{1}{2}$$

Hence

$$\frac{1}{(s+1)(s^2+1)} = \frac{\frac{1}{2}}{s+1} + \frac{-\frac{1}{2}s + \frac{1}{2}}{s^2+1}$$

Therefore

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= \frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t\end{aligned}$$

Method 2, using complex variables:

$$\frac{1}{(s+1)(s^2+1)} = \frac{1}{(s+1)(s+i)(s-i)} = \frac{A}{s+1} + \frac{B}{s+i} + \frac{C}{s-i}$$

$s = -1$ gives $A = \frac{1}{2}$ as above. $s = -i$ gives

$$\frac{1}{(-i+1)(-2i)} = \frac{1}{2(-1-i)} = \frac{-1}{2(1+i)} \times \frac{1-i}{1-i} = -\left(\frac{1-i}{4}\right) = -\frac{1}{4} + \frac{1}{4}i = B$$

and $s = i$ gives

$$\frac{1}{(i+1)(2i)} = -\frac{1}{2(1-i)} \times \frac{1+i}{1+i} = -\frac{1+i}{4} = -\frac{1}{4} - \frac{1}{4}i = C$$

Thus

$$\frac{1}{(s+1)(s^2+1)} = \frac{1}{2}\left(\frac{1}{s+1}\right) + \left(-\frac{1}{4} + \frac{1}{4}i\right)\left(\frac{1}{s+i}\right) + \left(-\frac{1}{4} - \frac{1}{4}i\right)\left(\frac{1}{s-i}\right)$$

So

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \left(-\frac{1}{4} + \frac{1}{4}i\right)\mathcal{L}^{-1}\left\{\frac{1}{s+i}\right\} + \left(-\frac{1}{4} - \frac{1}{4}i\right)\mathcal{L}^{-1}\left\{\frac{1}{s-i}\right\} \\ &= \frac{1}{2}e^{-t} + \left(-\frac{1}{4} + \frac{1}{4}i\right)e^{-it} + \left(-\frac{1}{4} - \frac{1}{4}i\right)e^{it} \\ &= \frac{1}{2}e^{-t} + \left(-\frac{1}{4} + \frac{1}{4}i\right)(\cos t - i\sin t) + \left(-\frac{1}{4} - \frac{1}{4}i\right)(\cos t + i\sin t) \\ &= \frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t\end{aligned}$$

SNB check $\frac{1}{(s+1)(s^2+1)}$, Is Laplace transform of $\frac{1}{2}\sin t - \frac{1}{2}\cos t + \frac{1}{2}e^{-t}$

2b (15 pts.) Find $\mathcal{L}^{-1}\left\{\frac{s+2}{s^2-3s+4}\right\}$.

Solution

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s+2}{s^2-3s+4}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+2}{s^2-3s+\frac{9}{4}+4-\frac{9}{4}}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{s+2}{\left(s-\frac{3}{2}\right)^2+\frac{7}{4}}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{s+2}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{s-\frac{3}{2}+\frac{7}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{s-\frac{3}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{7}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{s-\frac{3}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right\} + \sqrt{7} \mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{7}}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right\} \\
&= e^{\frac{3}{2}t} \cos\left(\frac{\sqrt{7}}{2}t\right) + \sqrt{7} e^{\frac{3}{2}t} \sin\left(\frac{\sqrt{7}}{2}t\right)
\end{aligned}$$

SNB check: $\frac{s+2}{s^2-3s+4}$, Is Laplace transform of $e^{\frac{3}{2}t} \left(\cos \frac{1}{2} \sqrt{7} t + \sqrt{7} \sin \frac{1}{2} \sqrt{7} t \right)$

3 (25 pts.) Find the first six non-zero terms in the series solution near $x = 0$ of the equation

$$y'' + 2xy' - 4y = 0$$

Be sure to give the recurrence relation. Indicate the two linearly independent solutions and give the first six nonzero terms of the solution.

Solution:

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y'(x) &= \sum_{n=1}^{\infty} a_n(n) x^{n-1} \\
y''(x) &= \sum_{n=2}^{\infty} a_n(n)(n-1) x^{n-2}
\end{aligned}$$

Substituting into the DE we have

$$\begin{aligned}
\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} + 2 \sum_{n=1}^{\infty} a_n(n)x^n - 4 \sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} + \sum_{n=1}^{\infty} a_n(2n-4)x^n - 4a_0 &= 0
\end{aligned}$$

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We now shift the first summation in the last equation above by letting $k = n - 2$ or $n = k + 2$.

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k + \sum_{n=1}^{\infty} a_n(2n-4)x^n - 4a_0 = 0$$

We write the first term of the first sum separately and replace k and n by m to get

$$\sum_{m=1}^{\infty} [a_{m+2}(m+2)(m+1) + a_m(2m-4)]x^m - 4a_0 + a_2(2)(1) = 0$$

Therefore

$$a_2 = 2a_0$$

and the recurrence relation is

$$a_{m+2}(m+2)(m+1) + a_m(2m-4) = 0 \quad m = 1, 2, \dots$$

or

$$a_{m+2} = \frac{2(2-m)}{(m+2)(m+1)} a_m \quad m = 1, 2, \dots$$

$$m = 1 \Rightarrow$$

$$a_3 = \frac{2(1)}{(3)(2)} a_1 = \frac{1}{3} a_1$$

$$m = 2 \Rightarrow$$

$$a_4 = 0$$

$$m = 3 \Rightarrow$$

$$a_5 = \frac{2(-1)}{(5)(4)} a_3 = \frac{-2}{(5)(4)(3)} a_1$$

$m = 4 \Rightarrow a_6 = 0$. In fact all of the even coefficients are zero except for a_0 and a_2 .

$$m = 5 \Rightarrow$$

$$a_7 = \frac{2(-3)}{(7)(6)} a_5 = \frac{(2)(2)(3)}{(7)(6)(5)(4)(3)} a_1$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

$$= a_0[1 + 2x] + a_1 \left[x + \frac{1}{3}x^3 + \frac{-2}{(5)(4)(3)}x^5 + \frac{(2)(2)(3)}{(7)(6)(5)(4)(3)}x^7 + \dots \right]$$

4 (25 pts.) Find the eigenvalues and eigenfunctions for

$$y'' + \lambda y = 0; \quad y(0) = 0, \quad y(1) = 0$$

Be sure to consider all values of λ .

Solution: The characteristic equation is:

$$p(r) = r^2 + \lambda = 0$$

$$\Rightarrow r = \sqrt{-\lambda}$$

There are three cases:

Case 1: $-\lambda = 0$

$\Rightarrow r = 0$ is a repeated root.

$$\Rightarrow y = c_1 + c_2 x$$

$$0 = y(0) = c_1$$

$$0 = y(1) = c_2 \Rightarrow c_2 = 0$$

$$\Rightarrow y \equiv 0 \text{ (trivial)}$$

Case 2: $-\lambda > 0$, that is $\lambda < 0$ Let $\lambda = -k^2$ where $k \neq 0$.

$$r = \pm k$$

$$\Rightarrow y = c_1 e^{kx} + c_2 e^{-kx}$$

$$0 = y(0) = c_1 + c_2$$

$$\Rightarrow c_2 = -c_1$$

$$0 = y(1) = c_1 e^k - c_1 e^{-k}$$

$$0 = c_1 [e^k - e^{-k}]$$

$$\Rightarrow c_1 = 0 \Rightarrow c_2 = 0 \Rightarrow y \equiv 0 \text{ (trivial)}$$

Case 3: $-\lambda < 0$, that is $\lambda > 0$ Let $\lambda = k^2$ where $k \neq 0$

$$\Rightarrow r = \pm ki$$

$$\Rightarrow y = c_1 \cos kx + c_2 \sin kx$$

$$0 = y(0) = c_1$$

$$0 = y(1) = c_2 \sin k$$

$$\sin k = 0 \Rightarrow k = n\pi, \quad n = 1, 2, 3, \dots$$

$$\lambda_n = k^2 = n^2 \pi^2$$

\Rightarrow The eigenvalues are: $\lambda_n = n^2 \pi^2$ where $n = 1, 2, 3, \dots$

and the eigenfunctions are: $y_n = c_n \sin(n\pi x)$

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Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$		
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	$n \geq 1$	$s > 0$
e^{at}	$\frac{1}{s-a}$		$s > a$
$\sin bt$	$\frac{b}{s^2 + b^2}$		$s > a$
$\cos bt$	$\frac{s}{s^2 + b^2}$		$s > a$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		