Name:____________________________________ Lecure Section ___ Recitation Section ____

Ma 221 **Exam II A** **Solutions**
08S

I pledge my honor that I have abided by the Stevens Honor System.

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You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1a ________

#1b ________

#1c ________

#2a ________

#2b ________

#3 __________

Total Score ________

Name:__________________________________ Lecure Section ___ Recitation Section ____

1. Consider the differential equation

$$y'' + 4y' + 5y = 2e^{-2t} + \cos t$$

1 a (8 **pts**.) Find the homogeneous solution of this equation.

Solution: The characteristic equation is

$$p(r) = r^2 + 4r + 5 = 0$$

Thus

$$r = \frac{-4 \pm \sqrt{16 - 4(1)(5)}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

Therefore

$$y_h = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t$$

1 b (26 **pts**.) Find a particular solution of this equation.

Solution: We first find a particular solution corresponding to $2e^{-2t}$. Since -2 is not a root of $p(r)$, we have using the formula derived in class that

$$y_{p_1} = \frac{Ke^{\alpha t}}{p(\alpha)} = \frac{2e^{-2t}}{p(-2)} = \frac{2e^{-2t}}{1}$$

We present two ways of finding a particular solution corresponding to $\cos t$.

Method 1 (Using complex variables)

$$y'' + 4y' + 5y = \cos t$$
$$v'' + 4v' + 5v = \sin t$$

Multiplying the second equation by i, adding the result to the first equation and letting $w = y + iv$ we have

$$w'' + 4w' + 5w = \cos t + i\sin t = e^{it}$$

Thus

$$w_{p_2} = \frac{e^{it}}{p(i)} = \frac{e^{it}}{4+4i} = \frac{1}{4}\left(\frac{e^{it}}{1+i}\right)$$

Since $y_{p_2} = \operatorname{Re} w_{p_2}$ we find the real part of w_{p_2}.

$$\begin{aligned} w_{p_2} &= \frac{1}{4}\left(\frac{e^{it}}{1+i}\right)\left(\frac{1-i}{1-i}\right) = \frac{1}{8}(1-i)e^{it} \\ &= \frac{1}{8}(1-i)(\cos t + i\sin t) \\ &= \frac{1}{8}(\cos t + \sin t + i\sin t - i\cos t) \end{aligned}$$

Thus

$$y_{p_2} = \operatorname{Re} w_p = \frac{1}{8}(\cos t + \sin t)$$

Method 2: Let

$$y_{p_2} = A\cos t + B\sin t$$

The

$$y'_{p_2} = -A\sin t + B\cos t$$
$$y''_{p_2} = -A\cos t - B\sin t$$

Substituting into the DE we have

$$-A\cos t - B\sin t - 4A\sin t + 4B\cos t + 5A\cos t + 5B\sin t = \cos t$$

or

$$4A + 4B = 1$$
$$-4A + 4B = 0$$

Thus $A = B = \frac{1}{8}$ and

$$y_{p_2} = \frac{1}{8}\cos t + \frac{1}{8}\sin t$$

as before.
Hence

$$y_p = y_{p_1} + y_{p_2} = 2e^{-2t} + \frac{1}{8}(\cos t + \sin t)$$

1 c (6 **pts**.) Give a general solution of this equation.

$$y = y_h + y_p = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t + 2e^{-2t} + \frac{1}{8}(\cos t + \sin t)$$

SNB check: $y'' + 4y' + 5y = 2e^{-2t} + \cos t$, Exact solution is:
$\left\{\frac{1}{8e^{2t}}\left((\cos t)e^{2t} + e^{2t}\sin t + 16\right) + C_{21}(\cos t)e^{-2t} - C_{22}(\sin t)e^{-2t}\right\}$

2 Consider the equation

$$t^2y'' - ty' + y = t \quad t > 0 \tag{$*$}$$

2 a (10 **pts**.) Find two linearly independent homogeneous solutions of this equation and verify that they are linearly independent by showing that their
Wronskian is not zero for $t > 0$.

Solution: This is an Euler's equation with $p = -1$ and $q = 1.$ Thus the indicial equation is

$$m^2 + (p-1)m + q = m^2 - 2m + 1 = (m-1)^2$$

We see the 1 is a repeated root, so that the homogeneous solution is

$$y_h = c_1 t + c_2 t\ln t$$

The Wronskian is

$$\begin{vmatrix} t & t\ln t \\ 1 & \ln t + 1 \end{vmatrix} = t \neq 0 \text{ for } t > 0$$

2b (25 **pts**.)) Find a general solution of

$$t^2y'' - ty' + y = t \quad t > 0$$

To find a particular solution we use the Method of Variation of Parameters. Let

$$y_p = v_1 t + v_2 t\ln t$$

The two equations for v_1' and v_2' are

$$v_1' t + v_2' t \ln t = 0$$
$$v_1' + v_2'(\ln t + 1) = \frac{t}{t^2} = \frac{1}{t}$$

Thus

$$v_1' = \frac{\begin{vmatrix} 0 & t\ln t \\ \frac{1}{t} & \ln t + 1 \end{vmatrix}}{\begin{vmatrix} t & t\ln t \\ 1 & \ln t + 1 \end{vmatrix}} = -\frac{\ln t}{t}$$

and

$$v_2' = \frac{\begin{vmatrix} t & 0 \\ 1 & \frac{1}{t} \end{vmatrix}}{\begin{vmatrix} t & t\ln t \\ 1 & \ln t + 1 \end{vmatrix}} = \frac{1}{t}$$

Thus

$$v_1 = -\int \left(\frac{\ln t}{t}\right) dt = -\frac{(\ln t)^2}{2}$$
$$v_2 = \int \frac{1}{t} dt = \ln t$$

and

$$y_p = -t\frac{(\ln t)^2}{2} + t(\ln t)^2 = \frac{t}{2}(\ln t)^2$$

Hence

$$y = y_h + y_p = c_1 t + c_2 t \ln t + \frac{t}{2}(\ln t)^2$$

3 (25 pts.) Find a general solution of

$$9y'' - 6y' + y = 9te^{\frac{t}{3}}$$

Solution: The characteristic equation is

$$9r^2 - 6r + 1 = (3r - 1)(3r - 1) = 0$$

Thus $r = 1/3$ is a repeated root and

$$y_h = c_1 e^{\frac{t}{3}} + c_2 t e^{\frac{t}{3}}$$

Given the form of the homogeneous solution, we see that if there were an term in $e^{\frac{t}{3}}$ on the right hand side, then we would seek a particular of the form $At^2 e^{\frac{t}{3}}$. However, since the term on the right hand side involves $te^{\frac{t}{3}}$ we let

$$y_p = At^3 e^{\frac{t}{3}}$$

Then

$$y_p' = 3At^2e^{\frac{t}{3}} + \frac{1}{3}At^3e^{\frac{t}{3}}$$

$$y_p'' = 6Ate^{\frac{t}{3}} + At^2e^{\frac{t}{3}} + At^2e^{\frac{t}{3}} + \frac{1}{9}At^3e^{\frac{t}{3}} = 6Ate^{\frac{t}{3}} + 2At^2e^{\frac{t}{3}} + \frac{1}{9}At^3e^{\frac{t}{3}}$$

Plugging into the DE we have

$$54Ate^{\frac{t}{3}} + 18At^2e^{\frac{t}{3}} + At^3e^{\frac{t}{3}} - 18At^2e^{\frac{t}{3}} - 2At^3e^{\frac{t}{3}} + At^3e^{\frac{t}{3}} = 9te^{\frac{t}{3}}$$

Thus

$$54A = 9 \Rightarrow A = \frac{1}{6}$$

and

$$y_p = \frac{1}{6}t^3e^{\frac{t}{3}}$$

Thus

$$y = y_h + y_p = c_1e^{\frac{t}{3}} + c_2te^{\frac{t}{3}} + \frac{1}{6}t^3e^{\frac{t}{3}}$$

SNB check: $9y'' - 6y' + y = 9te^{\frac{t}{3}}$, Exact solution is: $\left\{C_8e^{\frac{1}{3}t} + \frac{1}{6}t^3e^{\frac{1}{3}t} + C_9te^{\frac{1}{3}t}\right\}$