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Ma 221 Exam IIIB Solutions 08S

I pledge my honor that I have abided by the Stevens Honor System.
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You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1 ________

#2 ________

#3 ________

#4 ________

Total Score ________

Note: A table of Laplace Transforms is given at the end of the exam.

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1 $\left(25\ \textbf{pts.}\right)$ Use Laplace Transforms to solve

$$y'' + 2y' - 8y = e^t \quad y(0) = 0 \;\; y'(0) = 1$$

Solution: Taking the Laplace Transform of the DE leads to

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2s\mathcal{L}\{y\} - y(0) - 8\mathcal{L}\{y\} = \frac{1}{s-1}$$

So

$$(s^2 + 2s - 8)\mathcal{L}\{y\} = \frac{1}{s-1} + 1 = \frac{s}{s-1}$$

Therefore

$$\mathcal{L}\{y\} = \frac{s}{(s-1)(s+4)(s-2)}$$

Using partial fractions we have

$$\frac{s}{(s-1)(s+4)(s-2)} = \frac{A}{s-1} + \frac{B}{s+4} + \frac{C}{s-2}$$

Solving we have

$$A = \frac{1}{(5)(-1)} = -\frac{1}{5} \quad B = \frac{-4}{(-5)(-6)} = -\frac{2}{15} \quad C = \frac{2}{(1)(6)} = \frac{1}{3}$$

Therefore

$$\mathcal{L}\{y\} = \frac{s}{(s-1)(s+4)(s-2)} = -\frac{1}{5}\frac{1}{s-1} - \frac{2}{15}\frac{1}{s+4} + \frac{1}{3}\left(\frac{1}{s-2}\right)$$

and

$$y(t) = -\frac{1}{5}e^t - \frac{2}{15}e^{-4t} + \frac{1}{3}e^{2t}$$

SNB check: $\begin{array}{l} y'' + 2y' - 8y = e^t \\ y(0) = 0 \\ y'(0) = 1 \end{array}$, Exact solution is : $y(t) = -\frac{1}{5}e^t + \frac{1}{3}e^{2t} - \frac{2}{15}e^{-4t}$

2a $\left(15\ \textbf{pts.}\right)$ Find the partial fractions break down of

$$\frac{4s^2 - 4s + 30}{s(s^2 - 2s + 10)}$$

Solution:

$$\frac{4s^2 - 4s + 30}{s(s^2 - 2s + 10)} = \frac{A}{s} + \frac{Bs + C}{s^2 - 2s + 10}$$

Multiplying by s and setting $s = 0$ yields

$$\frac{30}{10} = 3 = A$$

Thus

$$\frac{4s^2 - 4s + 30}{s(s^2 - 2s + 10)} = \frac{3}{s} + \frac{Bs + C}{s^2 - 2s + 10}$$

Setting $s = 1$ and $s = -1$ we have

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$$\frac{4-4+30}{1(1-2+10)} = \frac{3}{1} + \frac{B+C}{9}$$
$$\frac{4+4+30}{-1(1+2+10)} = \frac{3}{-1} + \frac{-B+C}{13}$$

or

$$\frac{30}{9} = 3 + \frac{B+C}{9}$$
$$\frac{38}{-13} = -3 + \frac{-B+C}{13}$$

or

$$30 = 27 + B + C$$
$$-38 = -39 - B + C$$

Adding we have

$$-8 = -12 + 2C$$

so $C = 2.$ Thus $B = -C + 3 = 1$ and

$$\frac{4s^2-4s+30}{s(s^2-2s+10)} = \frac{3}{s} + \frac{s+2}{s^2-2s+10}$$

2b $\left(10\ \textbf{pts}.\right)$ Find

$$\mathcal{L}^{-1}\left\{\frac{4s^2-4s+30}{s(s^2-2s+10)}\right\}$$

Solution:

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{4s^2-4s+30}{s(s^2-2s+10)}\right\} &= \mathcal{L}^{-1}\left\{\frac{3}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{s+2}{s^2-2s+10}\right\} \\
&= 3 + \mathcal{L}^{-1}\left\{\frac{s+2}{(s-1)^2+9}\right\} \\
&= 3 + \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+9} + \frac{3}{(s-1)^2+9}\right\} \\
&= 3 + \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{(s-1)^2+9}\right\} \\
&= 3 + e^t\cos 3t + e^t \sin 3t
\end{aligned}$$

3 $\left(25\ \textbf{pts}.\right)$ Find the first six non-zero terms in the series solution near $x = 0$ of the equation

$$y'' - 3x^2 y = 0$$

Be sure to give the recurrence relation. Indicate the two linearly independent solutions and give the first *six* nonzero terms of the solution.
Solution:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} a_n (n) x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

Substituting into the DE we have

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} - 3 \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Shifting the first sum by letting $n - 2 = k + 2$ or $n = k + 4$ leads to

$$\sum_{k=-2}^{\infty} a_{k+4} (k+4)(k+3) x^{k+2} - 3 \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Replacing k and n by m we have

$$\sum_{m=-2}^{\infty} a_{m+4} (m+4)(m+3) x^{m+2} - 3 \sum_{m=0}^{\infty} a_m x^{m+2} = 0$$

or

$$a_2(2)(1) + a_3(3)(2)x + \sum_{m=0}^{\infty} [a_{m+4}(m+4)(m+3) - 3a_m] x^{m+2} = 0$$

Therefore $a_2 = a_3 = 0$ and the recurrence relation is

$$a_{m+4} = \frac{3}{(m+4)(m+3)} a_m \quad m = 0, 1, 2, \ldots$$

Hence

$$a_4 = \frac{3}{(4)(3)} a_0 \quad a_5 = \frac{3}{(5)(4)} a_1$$

$$a_6 = 0 \quad a_7 = 0$$

$$a_8 = \frac{3}{8(7)} a_4 = \frac{9}{8(7)(4)(3)} a_0$$

$$a_9 = \frac{3}{9(8)} a_5 = \frac{9}{9(8)(5)(4)} a_1$$

Thus

$$y(x) = a_0 \left[1 + \frac{3}{(4)(3)} x^4 + \frac{9}{8(7)(4)(3)} x^8 + \cdots \right] + a_1 \left[x + \frac{3}{(5)(4)} x^5 + \frac{9}{9(8)(5)(4)} x^9 + \cdots \right]$$

4 (25 **pts**.) Find the eigenvalues and eigenfunctions for

$$y'' + \lambda y = 0; \quad y(0) = 0, \quad y'(3) = 0$$

Be sure to consider all values of λ.

Solution:

Case I: $\lambda < 0$. Let $\lambda = -\alpha^2$ where $\alpha \neq 0$. Then the DE is

$$y'' - \alpha^2 y = 0$$

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which has the solutions $y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$. Also $y'(x) = c_1 \alpha e^{\alpha x} - c_2 \alpha e^{-\alpha x}$. The BCs imply

$$y(0) = c_1 + c_2 = 0$$

$$y'(3) = c_1 \alpha e^{3\alpha} - c_2 \alpha e^{-3\alpha}$$

The first equation implies that $c_1 = -c_2$ and the second equation then becomes

$$c_1 \alpha (e^{3\alpha} + e^{-3\alpha}) = 0$$

Thus $c_1 = 0$ and hence $c_2 = 0$ and therefore the only solution for this case is $y = 0$. There are no negative eigenvalues.

Case II: $\lambda = 0$. Then $y = Ax + B$. $y(0) = B = 0$ and $y'(3) = A = 0$. Thus zero is not an eigenvalue.

Case III: $\lambda > 0$. Let $\lambda = \beta^2$ where $\beta \neq 0$. The DE is $y'' + \beta^2 y = 0$ so

$$y(x) = c_1 \sin \beta x + c_2 \cos \beta x$$

The condition $y(0) = 0$ implies that $c_2 = 0$ so $y(x) = c_1 \sin \beta x$. The BC $y'(3) = 0$ leads to

$$c_1 \cos 3\beta = 0$$

and

$$3\beta = (2n+1)\frac{\pi}{2} \quad n = 0, 1, 2, \ldots$$

so the eigenvalues are

$$\lambda = \beta^2 = (2n+1)^2 \frac{\pi^2}{36} \quad n = 0, 1, 2, \ldots$$

The eigenfunctions are

$$y_n(x) = a_n \sin\left[(2n+1)\frac{\pi}{6} x \right] \quad n = 0, 1, 2, \ldots$$

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Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$		
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	$n \geq 1$	$s > 0$
e^{at}	$\frac{1}{s-a}$		$s > a$
$\sin bt$	$\frac{b}{s^2 + b^2}$		$s > 0$
$\cos bt$	$\frac{s}{s^2 + b^2}$		$s > 0$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		