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Ma 221 **Exam II A Solutions** **09S**

I pledge my honor that I have abided by the Stevens Honor System.

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You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1a ________

#1b ________

#1c ________

#2 _________

#3 _________

#4 _________

Total Score ________

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1. Consider the differential equation

$$y'' - 4y' - 12y = 2e^{-2t} + 2t^3 - t + 3$$

1 a (6 **pts**.) Find the homogeneous solution of this equation.

Solution: The characteristic equation is

$$p(r) = r^2 - 4r - 12 = (r-6)(r+2) = 0$$

Thus $r = 6, -2$ and

$$y_h = c_1 e^{6t} + c_2 e^{-2t}$$

1 b (25 **pts**.) Find a particular solution of this equation.

Solution: We first find a particular solution for $2e^{-2t}$. Since e^{-2t} is a homogeneous solution ($p(-2) = 0$) but te^{-2t} is not, since $p'(r) = 2r - 4$ and $p'(-2) = -8 \neq 0$ then

$$y_{p_1} = \frac{2te^{-2t}}{-8} = -\frac{1}{4}te^{-2t}$$

We now find a particular solution for $2t^3 - t + 3$. Let

$$y_{p_2} = At^3 + Bt^2 + Ct + D$$

Then

$$y'_{p_2} = 3At^2 + 2Bt + C$$
$$y''_{p_2} = 6At + 2B$$

Substituting into the DE yields

$$6At + 2B - 12At^2 - 8Bt - 4C - 12At^3 - 12Bt^2 - 12Ct - 12D = 2t^3 - t + 3$$

Thus we have

$$-12A = 2$$
$$-12A - 12B = 0$$
$$6A - 8B - 12C = -1$$
$$2B - 4C - 12D = 3$$

Hence $A = -\frac{1}{6}$, $B = -A = \frac{1}{6}$, $C = -\frac{1}{12}\left(-1 - 6\left(-\frac{1}{6}\right) + 8\left(\frac{1}{6}\right)\right) = -\frac{1}{9}$, and $D = -\frac{1}{12}\left(3 - 2\left(\frac{1}{6}\right) + 4\left(-\frac{1}{9}\right)\right) = -\frac{5}{27}$
and

$$y_{p_2} = -\frac{1}{6}t^3 + \frac{1}{6}t^2 - \frac{1}{9}t - \frac{5}{27}$$

Hence

$$y_p = y_{p_1} + y_{p_2} = -\frac{1}{4}te^{-2t} - \frac{1}{6}t^3 + \frac{1}{6}t^2 - \frac{1}{9}t - \frac{5}{27}$$

1 c (4 **pts**.) Give a general solution of this equation.

Solution:

$$y_g = y_h + y_{p_1} + y_{p_2} = c_1 e^{6t} + c_2 e^{-2t} - \frac{1}{4}te^{-2t} - \frac{1}{6}t^3 + \frac{1}{6}t^2 - \frac{1}{9}t - \frac{5}{27}$$

2 (25 **pts**.) Find a general solution of

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$$y'' - 2y' + y = \frac{e^t}{t^2+1}$$

Note: $\int \frac{dt}{t^2+1} = \arctan t + C$.

Solution: We first find the homogeneous solution. The characteristic equation is

$$p(r) = r^2 - 2r + 1 = (r-1)^2$$

so $r = 1$ is a repeated root. Thus

$$y_h = c_1 e^t + c_2 t e^t$$

We let

$$y_p = v_1 e^t + v_2 t e^t$$

and the two equations for v_1' and v_2' are

$$v_1' e^t + v_2' t e^t = 0$$
$$v_1' e^t + v_2'(te^t + e^t) = \frac{e^t}{t^2+1}$$

We may rewrite these equations as

$$v_1' + v_2' t = 0$$
$$v_1' + v_2'(t+1) = \frac{1}{t^2+1}$$

Hence

$$v_1' = \frac{\begin{vmatrix} 0 & t \\ \frac{1}{t^2+1} & t+1 \end{vmatrix}}{\begin{vmatrix} 1 & t \\ 1 & t+1 \end{vmatrix}} = -\frac{\frac{t}{t^2+1}}{1} = -\frac{t}{t^2+1}$$

$$v_2' = \frac{\begin{vmatrix} 1 & 0 \\ 1 & \frac{1}{t^2+1} \end{vmatrix}}{\begin{vmatrix} 1 & t \\ 1 & t+1 \end{vmatrix}} = \frac{1}{t^2+1}$$

Thus

$$v_1 = -\int \frac{t}{t^2+1} dt = -\frac{1}{2} \ln\left(t^2+1\right)$$
$$v_2 = \int \frac{dt}{t^2+1} = \arctan t$$

Therefore

$$y_p = v_1 e^t + v_2 t e^t = -\frac{1}{2} \ln\left(t^2+1\right) e^t + t e^t \arctan t$$

and

$$y_g = y_h + y_p = c_1 e^t + c_2 t e^t - \frac{1}{2} \ln\left(t^2+1\right) e^t + t e^t \arctan t$$

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3 (**25 pts**.) Find a general solution of

$$y'' - 2y' + 2y = e^t \sin t$$

Solution: The characteristic equation is

$$p(r) = r^2 - 2r + 2 = 0$$

so

$$r = \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2} = 1 \pm i$$

and

$$y_h = c_1 e^t \sin t + c_2 e^t \cos t$$

To find a particular solution we consider the companion equation

$$v'' - 2v' + 2v = e^t \cos t$$

Multiply the equation for y by i add it to the equation for v and let $w = v + iy$ to get

$$w'' - 2w' + 2w = e^t \cos t + ie^t \sin t = e^t(\cos t + i \sin t) = e^t e^{it} = e^{(1+i)t}$$

Since $p(1+i) = 0,$ and $p'(r) = 2r - 2$ so that $p'(1+i) = 2i$

$$w_p = \frac{te^{(1+i)t}}{2i}$$

We want y_p which is the imaginary part of w_p.

$$w_p = \frac{te^{(1+i)t}}{2i} \times \frac{i}{i} = -\frac{i}{2} te^t(\cos t + i \sin t)$$

Thus

$$y_p = \operatorname{Im} w_p = -\frac{1}{2} te^t \cos t$$

and

$$y_g = y_h + y_p = c_1 e^t \sin t + c_2 e^t \cos t - \frac{1}{2} te^t \cos t$$

4 (15 **pts**.) Solve

$$x^2 y'' + 2xy' + y = 0$$

Solution: This is an Euler equation with $p = 2$ and $q = 1.$ The indicial equation is

$$r^2 + (p-1)r + q = r^2 + r + 1 = 0$$

Hence

$$r = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$

This is the case of complex roots. For comples roots $a \pm bi$ the solution is

$$y_h = x^a[A \cos(b \ln x) + B \sin(b \ln x)].$$

Here $a = -\frac{1}{2}$ and $b = \frac{\sqrt{3}}{2}$ so

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$$y_h = x^{-\frac{1}{2}}\left[A\cos\left(\frac{\sqrt{3}}{2}\ln x\right)+B\sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right]$$