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Ma 221 Exam IIIA Solutions 09S

I pledge my honor that I have abided by the Stevens Honor System.

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You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1 ________

#2 ________

#3 ________

#4 ________

Total Score ________

Note: A table of Laplace Transforms is given at the end of the exam.

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1 $\left(25\ \textbf{pts}.\right)$ Use Laplace Transforms to solve

$$y'' - 10y' + 9y = 8t \quad y(0) = 1 \;\; y'(0) = 1$$

Solution: We take the Laplace Transform of both sides to get

$$\mathcal{L}\{y''\} - 10\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{8t\} = \frac{8}{s^2}$$

or letting $\mathcal{L}\{y\} = Y(s)$

$$s^2Y - sy(0) - y'(0) - 10(sY - y(0)) + 9Y = \frac{8}{s^2}$$

$\Rightarrow$

$$(s^2 - 10s + 9)Y = \frac{8}{s^2} + s + 1 - 10$$

$$Y = \frac{8}{s^2(s-9)(s-1)} + \frac{s-9}{(s-9)(s-1)} = \frac{8 + s^3 - 9s^2}{s^2(s-9)(s-1)}$$

We have to decompose this last fraction into partial fractions.

$$\frac{8 + s^3 - 9s^2}{s^2(s-9)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

Multiplying by s^2 and setting $s = 0$ gives

$$\frac{8}{9} = B$$

Multiplying by $s - 9$ and setting $s = 9$ gives

$$\frac{8 + 81(9) - 9(81)}{81(8)} = \frac{8}{81(8)} = \frac{1}{81} = C$$

Multiplying by $s - 1$ and setting $s = 1$ gives

$$\frac{8 + 1 - 9}{-8} = 0 = D$$

Thus we now have

$$\frac{8 + s^3 - 9s^2}{s^2(s-9)(s-1)} = \frac{A}{s} + \frac{\frac{8}{9}}{s^2} + \frac{\frac{1}{81}}{s-9}$$

Let $s = -1.$ Then

$$\frac{8 - 1 - 9}{(-10)(-2)} = -A + \frac{8}{9} - \frac{1}{810}$$

$$\frac{-2}{20} = -A + \frac{720 - 1}{810} = -A + \frac{719}{810}$$

$$A = \frac{1}{10} + \frac{719}{810} = \frac{81 + 719}{810} = \frac{800}{810} = \frac{80}{810}$$

Thus

$$Y(s) = \frac{8 + s^3 - 9s^2}{s^2(s-9)(s-1)} = \frac{1}{81(s-9)} + \frac{80}{81s} + \frac{8}{9s^2}$$

Hence

$$y(t) = \frac{1}{81}e^{9t} + \frac{80}{81} + \frac{8}{9}t$$

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$$
\begin{aligned}
y'' - 10y' + 9y &= 8t \\
y(0) &= 1 \\
y'(0) &= 1
\end{aligned}
$$

, Exact solution is: $\left\{\frac{8}{9}t + \frac{1}{81}e^{9t} + \frac{80}{81}\right\}$

2a (15 pts.) Use the definition of the Laplace transform to find $\mathcal{L}\{f(t)\}$ where

$$
f(t) = \begin{cases} 8 & 0 \le t \le 8 \\ \\ t & t \ge 8 \end{cases}
$$

Solution:

$$
\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)dt = \int_0^8 8e^{-st}dt + \int_8^\infty te^{-st}dt \\
&= -\frac{8}{s}e^{-st}|_0^8 + \lim_{R\to\infty}\left[\int_8^R te^{-st}dt\right] \quad \text{let } u = t \quad dv = e^{-st}dt \Rightarrow v = -\frac{e^{-st}}{s} \\
&= \frac{8}{s}[1 - e^{-8s}] + \lim_{R\to\infty}\left[-t\frac{e^{-st}}{s}|_8^R + \frac{1}{s}\int_8^R e^{-st}dt\right] \\
&= \frac{8}{s}[1 - e^{-8s}] + \lim_{R\to\infty}\left[-R\frac{e^{-sR}}{s} + \frac{8e^{-8s}}{s}\right] + \left(-\frac{1}{s^2}\right)\lim_{R\to\infty}[e^{-Rs} - e^{-8s}] \quad s > 0 \\
&= \frac{8}{s}[1 - e^{-8s}] - \frac{8e^{-8s}}{s} + \frac{e^{-8s}}{s^2} \quad s > 0
\end{aligned}
$$

2b (10 **pts.**) Find

$$
\mathcal{L}^{-1}\left\{\frac{1 - 3s}{s^2 + 8s + 21}\right\}
$$

Solution:

$$
\begin{aligned}
\frac{1 - 3s}{s^2 + 8s + 21} &= \frac{1 - 3s}{(s+4)^2 + 5} \\
&= \frac{1 - 3(s+4) + 12}{(s+4)^2 + 5} \\
&= \frac{13}{(s+4)^2 + 5} - 3\frac{s+4}{(s+4)^2 + 5} \\
&= \frac{13}{(s+4)^2 + \left(\sqrt{5}\right)^2} - 3\frac{s+4}{(s+4)^2 + \left(\sqrt{5}\right)^2} \\
&= \left(\frac{1}{\sqrt{5}}\right)\frac{13\sqrt{5}}{(s+4)^2 + \left(\sqrt{5}\right)^2} - 3\frac{(s+4)}{(s+4)^2 + \left(\sqrt{5}\right)^2}
\end{aligned}
$$

Thus

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$$\mathcal{L}^{-1}\left\{\frac{1-3s}{s^2+8s+21}\right\} = \left(\frac{13}{\sqrt{5}}\right)\mathcal{L}^{-1}\left\{\frac{\sqrt{5}}{(s+4)^2+\left(\sqrt{5}\right)^2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{(s+4)}{(s+4)^2+\left(\sqrt{5}\right)^2}\right\}$$

$$= \left(\frac{13}{\sqrt{5}}\right)e^{-4t}\sin\sqrt{5}\,t - 3e^{-4t}\cos\sqrt{5}\,t$$

3 $\left(25\right.$ **pts.**$\left.\right)$ Find the first <u>six</u> non-zero terms in the series solution near $x = 0$ of the equation

$$y'' - xy = 0$$

Be sure to give the recurrence relation. Indicate the two linearly independent solutions and give the first *six* nonzero terms of the solution.

Solution:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$$

Substituting into the DE we have

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

We shift the second sum. Let $k-2 = n+1$ so that $n = k-3$ and since $n = 0 \Rightarrow k = 3$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{k=3}^{\infty} a_{k-3}x^{k-2} = 0$$

We replace n and k by m to get

$$a_2(2)(1) + \sum_{m=3}^{\infty}[a_m(m)(m-1) - a_{m-3}]x^{m-2} = 0$$

Thus

$$a_2 = 0$$

$$a_m = \frac{a_{m-3}}{(m)(m-1)} \quad m = 3, 4, 5, \ldots$$

$$\begin{aligned}
a_3 &= \frac{a_0}{3(2)} \\
a_4 &= \frac{a_1}{4(3)} \\
a_5 &= 0 \\
a_6 &= \frac{a_3}{6(5)} = \frac{a_0}{6(5)(3)(2)} \\
a_7 &= \frac{a_4}{7(6)} = \frac{a_1}{7(6)(4)(3)} \\
a_8 &= 0
\end{aligned}$$

Hence

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \\
&= a_0\left[1 + \frac{1}{3(2)}x^3 + \frac{1}{6(5)(3)(2)}x^6 + \cdots\right] + a_1\left[x + \frac{1}{4(3)}x^4 + \frac{1}{7(6)(4)(3)}x^7 + \cdots\right]
\end{aligned}$$

4 (25 **pts**.) Find the eigenvalues and eigenfunctions for

$$y'' + \lambda y = 0; \quad y(0) = 0, \quad y(2\pi) = 0$$

Be sure to consider all values of λ.

Solution: We consider three cases.

1. $\lambda < 0.$ Let $\lambda = -\alpha^2$ where $\alpha \neq 0$ so the DE becomes $y'' - \alpha^2 y = 0$ and $y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}.$ Then

$$\begin{aligned}
y(0) &= c_1 + c_2 = 0 \\
y(2\pi) &= c_1 e^{\alpha\pi} + c_2 e^{-\alpha\pi} = 0
\end{aligned}$$

Since the first equation implies $c_1 = -c_2$ the second equation implies that

$$c_1(e^{\alpha\pi} - e^{-\alpha\pi}) = 0$$

and we see that $c_1 = c_2 = 0.$ Thus the only solution for $\lambda < 0$ is $y = 0.$

2. $\lambda = 0$ Then $y = c_1 x + c_2.$ $y(0) = c_2 = 0.$ $y(2\pi) = 2c_1\pi = 0,$ so $c_1 = 0.$ Thus the only solution for $\lambda = 0$ is $y = 0.$

3. $\lambda > 0.$ Let $\lambda = \beta^2$ where $\beta \neq 0$ so the DE becomes $y'' + \beta^2 y = 0.$ The solution is $y = c_1 \sin\beta x + c_2 \cos\beta x.$ Since $y(0) = c_2 = 0,$ we see that $y = c_1 \sin\beta x$

$$y(2\pi) = c_1 \sin 2\pi\beta = 0$$

$\Rightarrow 2\pi\beta = n\pi, \;\; n = 1,2,3,\ldots$ and

$$\beta = \frac{n}{2} \quad n = 1,2,3,\ldots$$

Hence the eigenvalues are $\lambda = \beta^2 = \frac{n^2}{4}, \;\; n = 1,2,3,\ldots$ and the eigenfunctions are

$$y_n(x) = a_n \sin\left(\frac{nx}{2}\right) \quad n = 1,2,3,\ldots$$

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Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$		
$\dfrac{t^{n-1}}{(n-1)!}$	$\dfrac{1}{s^{n}}$	$n \geq 1$	$s > 0$
e^{at}	$\dfrac{1}{s-a}$		$s > a$
$\sin bt$	$\dfrac{b}{s^2 + b^2}$		$s > 0$
$\cos bt$	$\dfrac{s}{s^2 + b^2}$		$s > 0$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		