Ma 221 Exam IA Solutions 10S

Solve:

1 [25 **pts**.]

$$1+y^2+2(t+1)y\frac{dy}{dt} = 0,\ \ y(0) = 1$$

Solution: We write the equation as

$$\left(1+y^2\right)dt+2(t+1)ydy = 0$$

The $M = 1+y^2$ and $N = 2(t+1)y$ and

$$M_y = 2y = N_t$$

Hence the equation is exact and there exists a function $f(t,y)$ such that

$$f_t = M \text{ and } f_y = N$$

So

$$f_t = 1+y^2 \quad \Rightarrow \quad f = t+ty^2+g(y)$$

Also

$$f_y = 2ty+g'(y) = N = 2ty+2y$$

Therefore

$$g(y) = y^2+C$$

and

$$f = t+ty^2+y^2+C$$

and the solution is given by

$$t+ty^2+y^2 = k$$

$y(0) = 1$ implies

$$1 = k$$

so

$$t+ty^2+y^2 = 1$$

or

$$y = \sqrt{\frac{1-t}{1+t}}$$

Alternate Solution: We can also observe that the equation is separable. We write the equation a little differently and then integrate.

$$1+y^2+2(t+1)y\frac{dy}{dt} = 0$$

becomes

$$
\begin{aligned}
2(t+1)y\frac{dy}{dt} &= -\left(1+y^2\right) \\
\frac{2y}{\left(1+y^2\right)}\frac{dy}{dt} &= -\frac{1}{(t+1)} \\
\int \frac{2y}{\left(1+y^2\right)}dy &= \int -\frac{1}{(t+1)}dt
\end{aligned}
$$

The solution is given implicitly by

$$\ln\left(1+y^2\right) = -\ln|t+1| + C$$

Since $y(0) = 1$,

$$\ln(2) = -\ln(1) + C = C$$

Finally the solution is

$$
\begin{aligned}
\ln\left(1+y^2\right) &= -\ln|t+1| + \ln 2 \\
&= \ln\frac{2}{|t+1|}
\end{aligned}
$$

or

$$\left(1+y^2\right) = \frac{2}{|t+1|}.$$

To obtain the form of the first solution, one must observe that the absolute value can be dropped because the initial condition and continuity restricts us to $t+1 > 0$. Then it's just a little algebra.

$$y^2 = \frac{2}{t+1} - 1 = \frac{2-(t+1)}{t+1} = \frac{1-t}{t+1}.$$

2 [25 **pts**.]

$$ty' + 4y = 6t^2 \quad y(1) = 3 \quad t > 0$$

Solution: This equation is first order linear and may be written as

$$y + \frac{4}{t}y' = 6t$$

We multiply the DE by $e^{\int P(t)dt} = e^{\int \frac{4}{t}dt} = e^{4\ln t} = t^4$ and get

$$t^4y' + 4t^3y = 6t^5$$

or

$$\frac{d}{dt}\left(t^4y\right) = 6t^5$$

Hence

$$t^4y = t^6 + c$$

and

$$y = t^2 + \frac{c}{t^4}$$

The initial condition yields

$$3 = 1 + c \text{ or } c = 2$$

so

$$y = t^2 + \frac{2}{t^4}$$

3 [25 **points**]

$$(2y - \sin y)y' + t = \sin t \quad y(0) = 1$$

Solution: We rewrite the equation as

$$(2y - \sin y)dy + (t - \sin t)dt = 0$$

which is separable. Integrating we have

$$y^2 + \cos y + \frac{t^2}{2} + \cos t = c$$

The initial condition implies

$$1 + \cos 1 + 1 = c$$

so

$$y^2 + \cos y + \frac{t^2}{2} + \cos t = 2 + \cos 1$$

4 $\left[25\ \mathbf{pts}.\right]$

$$y' = 2t^{-1} + e^{-y} \quad y(1) = 0$$

Solution: Rewrite the equation as

$$e^y y' - \frac{2}{t}e^y = 1$$

Let $z = e^y$. Then $z' = e^y y'$ and the DE becomes

$$z' - \frac{2}{t}z = 1$$

This is first order linear in z. Multiply the DE by $e^{-\int \frac{2}{t}dt} = e^{-2\ln t} = t^{-2}$ to get

$$t^{-2}z - 2t^{-3}z = t^{-2}$$

or

$$\left(t^{-2}z\right)' = t^{-2}$$

Hence

$$t^{-2}z = -t^{-1} + c$$

or

$$z = e^y = -t + ct^2$$

The initial condition implies

$$1 = -1 + c \text{ or } c = 2$$

so

$$e^y = -t + 2t^2$$

or

$$y = \ln\left(2t^2 - t\right)$$