Ma 221 Exam IB Solutions 10S

Solve:

1 [25 **pts**.]

$$\left(2+y^2\right)+2(x-1)y\frac{dy}{dx} = 0, \qquad y(2) = 1.$$

Solution: We write the equation as

$$\left(2+y^2\right)dx+2(x-1)ydy = 0$$

The $M = 2+y^2$ and $N = 2(x-1)y$ and

$$M_y = 2y = N_t$$

Hence the equation is exact and there exists a function $f(t,y)$ such that

$$f_x = M \text{ and } f_y = N$$

So

$$f_x = 2+y^2 \quad \Rightarrow \quad f = 2x+xy^2+g(y)$$

Also

$$f_y = 2xy+g'(y) = N = 2xy-2y$$

Therefore

$$g(y) = -y^2+C$$

and

$$f = 2x+xy^2-y^2+C$$

and the solution is given by

$$2x+xy^2-y^2 = k$$

$y(2) = 1$ implies

$$5 = k$$

so

$$2x+xy^2-y^2 = 5$$

or

$$y = \sqrt{\frac{5-2x}{x-1}}$$

Alternate Solution: We can also observe that the equation is separable. We write the equation a little differently and then integrate.

$$\left(2+y^2\right)+2(x-1)y\frac{dy}{dx} = 0$$

becomes

$$2(x-1)y\frac{dy}{dx} = -\left(2+y^2\right)$$
$$\frac{2y}{\left(2+y^2\right)}\frac{dy}{dx} = -\frac{1}{(x-1)}$$
$$\int \frac{2y}{\left(2+y^2\right)}dy = \int -\frac{1}{(x-1)}dx$$

The solution is given implicitly by

$$\ln\left(2+y^2\right) = -\ln|x-1| + C$$

Since $y(2) = 1$,

$$\ln(3) = -\ln(1) + C = C$$

Finally the solution is

$$\ln\left(2+y^2\right) = -\ln|x-1| + \ln 3$$
$$= \ln\frac{3}{|x-1|}$$

or

$$\left(2+y^2\right) = \frac{3}{|x-1|}.$$

To obtain the form of the first solution, one must observe that the absolute value can be dropped because the initial condition and continuity restricts us to $x-1 > 0$. Then it's just a little algebra.

$$y^2 = \frac{3}{x-1} - 2 = \frac{3-2(x-1)}{x-1} = \frac{5-2x}{x-1}.$$

2 [25 **pts**.]

$$ty' + 5y = 7t^2, \qquad y(1) = 3$$

Solution: This equation is first order linear and may be written as

$$y + \frac{5}{t}y' = 7t$$

We multiply the DE by $e^{\int P(t)dt} = e^{\int \frac{5}{t}dt} = e^{5\ln t} = t^5$ and get

$$t^5y' + 5t4y = 7t^6$$

or

$$\frac{d}{dt}(t^5y) = 7t^6$$

Hence

$$t^5y = t^7 + c$$

and

$$y = t^2 + \frac{c}{t^5}$$

The initial condition yields

$$3 = 1 + c \text{ or } c = 2$$

so

$$y = t^2 + \frac{2}{t^5}$$

3 [25 **points**]

$$(2y - \cos y)y' + 2t = \cos t, \qquad y(0) = \pi$$

Solution: We rewrite the equation as

$$(2y - \cos y)dy + (2t - \cos t)dt = 0$$

$$(2y - \cos y)dy = (\cos t - 2t)dt$$

which is separable. Integrating we have

$$y^2 - \sin y = \sin t - t^2 + c$$

The initial condition implies

$$\pi^2 - \sin \pi = \sin 0 - 0^2 + c$$

$$c = \pi^2$$

so

$$y^2 - \sin y + t^2 - \sin t = \pi^2$$

4 $\left[\text{25 } \textbf{pts}.\right]$

$$\frac{dy}{dx} = 3x^{-1} + 2e^{-y}, \qquad y(1) = 0$$

Solution: Rewrite the equation as

$$e^y y' - \frac{3}{x}e^y = 2$$

Let $z = e^y$. Then $z' = e^y y'$ and the DE becomes

$$z' - \frac{3}{x}z = 2$$

This is first order linear in z. Multiply the DE by $e^{-\int \frac{3}{x}dx} = e^{-3\ln x} = x^{-3}$ to get

$$x^{-3}z - 3x^{-4}z = 2x^{-3}$$

or

$$\left(x^{-3}z\right)' = 2x^{-3}$$

Hence

$$x^{-3}z = -t^{-2} + c$$

or

$$z = e^y = -x + cx^3$$

The initial condition implies

$$1 = -1 + c \text{ or } c = 2$$

so

$$e^y = -x + 2x^3$$

or

$$y = \ln\left(2x^3 - x\right)$$