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Ma 221 **Exam II A Solutions** **10S**

I pledge my honor that I have abided by the Stevens Honor System.

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You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Note: A table of selected integrals appears on the last page of this exam.

Score on Problem #1a ________

#1b ________

#1c ________

#2a ________

#2b ________

#2c ________

#3 _________

Total Score ________

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1. (**30 pts**. **total**) Consider the differential equation

$$y'' - y' - 2y = 2e^{2t} - 20\sin 2t$$

1 a (6 **pts**.) Find the homogeneous solution of this equation.

Solution: The characteristic equation is

$$p(r) = r^2 - r - 2 = (r-2)(r+1) = 0$$

Thus the roots are $r = 2, -1$ and

$$y_h = c_1 e^{-t} + c_2 e^{2t}$$

1 b (20 **pts**.) Find a particular solution of this equation.

Solution: We find y_{p_1} for $2e^{2t}$ and y_{p_2} for $-20\sin 2t$. Since $p(2) = 0$, e^{2t} is a homogeneous solution. $p'(r) = 2r - 1$ so $p'(2) = 3 \neq 0$. Thus

$$y_{p_1} = \frac{Kte^{\alpha t}}{p'(\alpha)} = \frac{2te^{2t}}{3}$$

There are two ways to find y_{p_2}.

Method I Using Complex Variables: We consider two equations

$$y'' - y' - 2y = -20\sin 2t$$
$$v'' - v' - 2v = -20\cos 2t$$

Multiply the first equation by i and add it to the second equation. Letting $w = v + iy$ we have

$$w'' - w' - 2w = -20(\cos 2t + i\sin 2t) = -20e^{2it}$$

Since $p(2i) = (2i)^2 - 2i - 2 = -6 - 2i \neq 0$

$$w_p = \frac{Ke^{\alpha t}}{p(\alpha)} = \frac{10e^{2it}}{3+i}$$

y_{p_2} is the imaginary part of w_p.

$$w_p = \frac{10e^{2it}}{3+i}\left(\frac{3-i}{3-i}\right) = \frac{10(3-i)(\cos 2t + i\sin 2t)}{10}$$
$$= (3\cos 2t + \sin 2t) + i(-\cos 2t + 3\sin 2t)$$

Thus

$$y_{p_2} = 3\sin 2t - \cos 2t$$

Method II Without Using Complex Variables: Let

$$y_{p_2} = A\sin 2t + B\cos 2t$$
$$y'_{p_2} = 2A\cos 2t - 2B\sin 2t$$
$$y''_{p_2} = -4A\sin 2t - 4b\cos 2t$$

Substituting into

$$y'' - y' - 2y = -20\sin 2t$$

we have

$$-4A\sin 2t - 4B\cos 2t - 2A\cos 2t + 2B\sin 2t - 2A\sin 2t - 2B\cos 2t = -20\sin 2t$$

or

$$[-6A+2B]\sin 2t + [-2A-6B]\cos 2t = -20\sin 2t$$

Hence

$$-6A+2B = -20$$
$$-2A-6B = 0$$

From the second equation $A = -3B$ and from the first equation $B = -1$ and then $A = 3.$ Again

$$y_{p_2} = 3\sin 2t - \cos 2t$$

$$y_p = y_{p_1} + y_{p_2} = \frac{2te^{2t}}{3} + 3\sin 2t - \cos 2t$$

1 c (4 **pts**.) Give a general solution of this equation.

Solution:

$$y = y_h + y_p = c_1e^{-t} + c_2e^{2t} + \frac{2te^{2t}}{3} + 3\sin 2t - \cos 2t$$

SNB Check: $y'' - y' - 2y = 2e^{2t} - 20\sin 2t$, Exact solution is:
$\left\{3\sin 2t - \cos 2t + C_2e^{-t} + C_3e^{2t} + \frac{2}{3}te^{2t} - \frac{2}{9}e^{2t}\right\}$

2 (40 **pts**. **total**) Consider the Initial Value Problem

$$t^2y'' - ty' + y = t \quad y(1) = 1 \quad y'(1) = 4 \quad t > 0$$

2a (**5 pts**.) Find a homogeneous solution to this differential equation.

Solution: We first find the homogeneous solution to this Euler equation. The indicial equation is, since $p = -1$ and $q = 1$

$$m^2 + (p-1)m + q = m^2 - 2m + 1 = (m-1)^2$$

Thus $m = 1$ is a repeated root and

$$y_h = c_1t + c_2t\ln t$$

2b (**25 pts**.) Find a particular solution of this differential equation.

Solution: To find a particular solution we use the Method of Variations of Parameters with $y_1 = t$ and $y_2 = t\ln t.$ Then

$$y_p = v_1y_1 + v_2y_2 = tv_1 + t\ln t v_2$$

and the two equations for v_1' and v_2', namely

$$v_1'y_1 + v_2'y_2 = 0$$
$$v_1'y_1' + v_2'y_2' = \frac{f}{a}$$

become

$$v_1't + v_2't\ln t = 0$$
$$v_1' + v_2'(\ln t + 1) = \frac{t}{t^2} = \frac{1}{t}$$

$$W[t, t\ln t] = \begin{vmatrix} t & t\ln t \\ 1 & \ln t + 1 \end{vmatrix} = t \neq 0 \text{ since } t > 0$$

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Hence

$$v_1' = \frac{\begin{vmatrix} 0 & t\ln t \\ \frac{1}{t} & \ln t + 1 \end{vmatrix}}{W[t, t\ln t]} = -\frac{\ln t}{t}$$

$$v_2' = \frac{\begin{vmatrix} t & 0 \\ 1 & \frac{1}{t} \end{vmatrix}}{W[t, t\ln t]} = \frac{1}{t}$$

$$v_1 = \int\left(-\frac{\ln t}{t}\right)dt = -\frac{(\ln t)^2}{2}$$

$$v_2 = \int \frac{1}{t}dt = \ln t$$

Therefore

$$y_p = y_p = tv_1 + t\ln tv_2 = -\frac{t(\ln t)^2}{2} + t(\ln t)^2 = \frac{t(\ln t)^2}{2}$$

2c (10 pts.) Find the solution to this Initial Value Problem given above.

$$y = y_h + y_p = c_1 t + c_2 t\ln t + \frac{t(\ln t)^2}{2}$$

$$\begin{aligned} y' &= c_1 + c_2(\ln t + 1) + \frac{(\ln t)^2}{2} + \frac{2t\ln t\left(\frac{1}{t}\right)}{2} \\ &= c_1 + c_2(\ln t + 1) + \frac{(\ln t)^2}{2} + \ln t \end{aligned}$$

$$y(1) = c_1 = 1$$
$$y'(1) = c_1 + c_2 = 4$$

Thus $c_2 = 3$ and

$$y = t + 3t\ln t + \frac{t(\ln t)^2}{2}$$

3 (30 pts.) Find a general solution of

$$y'' + y' = 6x^2$$

Solution: The characteristic equation is

$$p(r) = r^2 + r = r(r+1)$$

Thus the roots are $r = 0, -1$ and

$$y_h = c_1 e^{0x} + c_2 e^{-x} = c_1 + c_2 e^{-x}$$

To find a particular solution we note that there is no y term in the DE. Hence we let

$$y_p = Ax + Bx^2 + Cx^3$$

Then

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$$y_p' = A + 2Bx + 3Cx^2$$
$$y_p'' = 2B + 6Cx$$

The DE implies

$$2B + 6Cx + A + 2Bx + 3Cx^2 = 6x^2$$

Hence

$$C = 2$$
$$2B + A = 0$$
$$6C + 2B = 0$$

$B = -6,\ \ A = 12$ and

$$y_p = 12x - 6x^2 + 2x^3$$
$$y = y_h + y_p = c_1 + c_2 e^{-x} + 12x - 6x^2 + 2x^3$$

SNB check $y'' + y' = 6x^2$, Exact solution is: $\left\{12x - C_2 + C_3 e^{-x} - 6x^2 + 2x^3 - 12\right\}$