1. (30 pts. total) Consider the differential equation

\[ y'' + y' - 2y = 2e^t - 20 \sin 2t \]

1a (6 pts.) Find the homogeneous solution of this equation.

Solution: The characteristic equation is

\[ p(r) = r^2 + r - 2 = (r + 2)(r - 1) = 0 \]

Thus the roots are \( r = -2, 1 \) and

\[ y_h = c_1 e^t + c_2 e^{-2t} \]

1b (20 pts.) Find a particular solution of this equation.

Solution: We find \( y_{p_1} \) for \( 2e^t \) and \( y_{p_2} \) for \(-20 \sin 2t\). Since \( p(1) = 0 \), \( e^t \) is a homogeneous solution.

\[ p'(r) = 2r + 1 \] so \( p'(1) = 3 \neq 0 \). Thus

\[ y_{p_1} = \frac{Kte^t}{p'(1)} = \frac{2te^t}{3} \]

There are two ways to find \( y_{p_2} \).

Method I Using Complex Variables: We consider two equations

\[ y'' + y' - 2y = -20 \sin 2t \]
\[ v'' + v' - 2v = -20 \cos 2t \]

Multiply the first equation by \( i \) and add it to the second equation. Letting \( w = v + iy \) we have

\[ w'' - w' - 2w = -20(\cos 2t + i \sin 2t) = -20e^{2it} \]

Since \( p(2i) = (2i)^2 + 2i - 2 = -6 + 2i \neq 0 \)

\[ w_p = \frac{Ke^{2it}}{p(2i)} = \frac{10e^{2it}}{3 - i} \]

\( y_{p_2} \) is the imaginary part of \( w_p \).

\[ w_p = \frac{10e^{2it}}{3 - i} \left( \frac{3 + i}{3 + i} \right) = \frac{10(3 + i)(\cos 2t + i \sin 2t)}{10} = (3 \cos 2t - \sin 2t) + i(\cos 2t + 3 \sin 2t) \]

Thus

\[ y_{p_2} = 3 \sin 2t + \cos 2t \]

Method II Without Using Complex Variables: Let

\[ y_{p_2} = A \sin 2t + B \cos 2t \]
\[ y'_{p_2} = 2A \cos 2t - 2B \sin 2t \]
\[ y''_{p_2} = -4A \sin 2t - 4B \cos 2t \]
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Substituting into
\[ y'' + y' - 2y = -20 \sin 2t \]
we have
\[ -4A \sin 2t - 4B \cos 2t + 2A \cos 2t - 2B \sin 2t - 2A \sin 2t - 2B \cos 2t = -20 \sin 2t \]
or
\[ [-6A - 2B] \sin 2t + [2A - 6B] \cos 2t = -20 \sin 2t \]
Hence
\[ -6A - 2B = -20 \]
\[ 2A - 6B = 0 \]
From the second equation \( A = 3B \) and from the first equation \( B = 1 \) and then \( A = 3 \). Again
\[ y_{p_2} = 3 \sin 2t + \cos 2t \]
\[ y_p = y_{p_1} + y_{p_2} = \frac{2te^{2t}}{3} + 3 \sin 2t + \cos 2t \]

1 c (4 pts.) Give a general solution of this equation.
Solution:
\[ y = y_h + y_p = c_1 e^t + c_2 e^{-2t} + \frac{2te^{2t}}{3} + 3 \sin 2t + \cos 2t \]

SNB Check: \( y'' + y' - 2y = 2e^t - 20 \sin 2t \), Exact solution is:
\[ C_1 e^t - \frac{2}{9} e^t + \frac{2}{3} te^t + \cos 2t + 3 \sin 2t + C_2 e^{-2t} \]

2 (40 pts. total) Consider the Initial Value Problem
\[ t^2 y'' - 3ty' + 4y = 2t^2 \quad y(1) = 1 \quad y'(1) = 4, y > 0. \]

2a (5 pts.) Find a homogeneous solution to this differential equation.
Solution: We first find the homogeneous solution to this Euler equation. The indicial equation is, since \( p = -3 \) and \( q = 4 \)
\[ m^2 + (p - 1)m + q = m^2 - 4m + 4 = (m - 2)^2 \]
Thus \( m = 1 \) is a repeated root and
\[ y_h = c_1 t^2 + c_2 t^2 \ln t \]

2b (25 pts.) Find a particular solution of this differential equation.
Solution: To find a particular solution we use the Method of Variations of Parameters with \( y_1 = t^2 \) and \( y_2 = t^2 \ln t \). Then
\[ y_p = v_1 y_1 + v_2 y_2 = t^2 v_1 + \left(t^2 \ln t\right)v_2 \]
and the two equations for \( v_1' \) and \( v_2' \), namely
\[ v_1'y_1 + v_2'y_2 = 0 \]
\[ v_1'y_1 + v_2'y_2 = \frac{f}{a} \]
become
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\[ t^2 v_1' + (t^2 \ln t)v_2' = 0 \]
\[ 2tv_1' + (2t \ln t + t)v_2' = \frac{2t^2}{t^2} = 2 \]

\[ W[t, t \ln t] = \begin{vmatrix} t^2 & t^2 \ln t \\ 2t & 2t \ln t + t \end{vmatrix} = t^3 \neq 0 \text{ since } t > 0 \]

Hence

\[ v_1' = -\frac{2t^2 \ln t}{t^3} = -\frac{2 \ln t}{t} \]
\[ v_2' = \frac{2t}{t^3} = \frac{2}{t} \]
\[ v_1 = \int \left( -\frac{2 \ln t}{t} \right) dt = -(\ln t)^2 \]
\[ v_2 = \int \frac{2}{t} dt = 2 \ln t \]

Therefore

\[ y_p = y_p = t^2 v_1 + (t^2 \ln t)v_2 = -(\ln t)^2 + 2t^2(\ln t)^2 = t^2(\ln t)^2 \]

2c (10 pts.) Find the solution to this Initial Value Problem given above.

\[ y'' - 2y' = 12x^2 \]

Solution: The characteristic equation is

\[ p(r) = r^2 - 2r = r(r - 2) \]

Thus the roots are \( r = 0, 2 \) and

\[ y_h = c_1 e^{0x} + c_2 e^{2x} = c_1 + c_2 e^{2x} \]

To find a particular solution we note that there is no \( y \) term in the DE. Hence we let
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\[ y_p = Ax + Bx^2 + Cx^3 \]
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Then
\[ y_p' = A + 2Bx + 3Cx^2 \]
\[ y_p'' = 2B + 6Cx \]
The DE implies
\[ 2B + 6Cx - 2A - 4Bx - 6Cx^2 = 12x^2 \]
Hence
\[ C = -2 \]
\[ 2B - 2A = 0 \]
\[ 6C - 4B = 0 \]
\[ B = A = -3 \]
and
\[ y_p = -3x - 3x^2 - 2x^3 \]
\[ y = y_h + y_p = c_1 + c_2 e^{2x} - 3x - 3x^2 - 2x^3 \]
SNB check \( y'' - 2y' = 12x^2 \), Exact solution is: \( C_2 e^{2x} - 3x - 3x^2 - 2x^3 - \frac{1}{2} C_1 - \frac{3}{2} \)