Ma 221 Exam II B Solutions 10S

1. (**30 pts**. **total**) Consider the differential equation

$$y'' + y' - 2y = 2e^t - 20\sin 2t$$

1 a (6 **pts**.) Find the homogeneous solution of this equation.

Solution: The characteristic equation is

$$p(r) = r^2 + r - 2 = (r+2)(r-1) = 0$$

Thus the roots are $r = -2, 1$ and

$$y_h = c_1 e^t + c_2 e^{-2t}$$

1 b (20 **pts**.) Find a particular solution of this equation.

Solution: We find y_{p_1} for $2e^t$ and y_{p_2} for $-20\sin 2t$. Since $p(1) = 0$, e^t is a homogeneous solution. $p'(r) = 2r + 1$ so $p'(1) = 3 \neq 0$. Thus

$$y_{p_1} = \frac{Kte^{\alpha t}}{p'(\alpha)} = \frac{2te^t}{3}$$

There are two ways to find y_{p_2}.

Method I Using Complex Variables: We consider two equations

$$y'' + y' - 2y = -20\sin 2t$$
$$v'' + v' - 2v = -20\cos 2t$$

Multiply the first equation by i and add it to the second equation. Letting $w = v + iy$ we have

$$w'' - w' - 2w = -20(\cos 2t + i\sin 2t) = -20e^{2it}$$

Since $p(2i) = (2i)^2 + 2i - 2 = -6 + 2i \neq 0$

$$w_p = \frac{Ke^{\alpha t}}{p(\alpha)} = \frac{10e^{2it}}{3-i}$$

y_{p_2} is the imaginary part of w_p.

$$w_p = \frac{10e^{2it}}{3-i}\left(\frac{3+i}{3+i}\right) = \frac{10(3+i)(\cos 2t + i\sin 2t)}{10}$$
$$= (3\cos 2t - \sin 2t) + i(\cos 2t + 3\sin 2t)$$

Thus

$$y_{p_2} = 3\sin 2t + \cos 2t$$

Method II Without Using Complex Variables: Let

$$y_{p_2} = A\sin 2t + B\cos 2t$$
$$y'_{p_2} = 2A\cos 2t - 2B\sin 2t$$
$$y''_{p_2} = -4A\sin 2t - 4b\cos 2t$$

Substituting into

$$y'' + y' - 2y = -20\sin 2t$$

we have

$$-4A\sin 2t - 4B\cos 2t + 2A\cos 2t - 2B\sin 2t - 2A\sin 2t - 2B\cos 2t = -20\sin 2t$$

or

$$[-6A - 2B]\sin 2t + [2A - 6B]\cos 2t = -20\sin 2t$$

Hence

$$-6A - 2B = -20$$
$$2A - 6B = 0$$

From the second equation $A = 3B$ and from the first equation $B = 1$ and then $A = 3.$ Again

$$y_{p_2} = 3\sin 2t + \cos 2t$$

$$y_p = y_{p_1} + y_{p_2} = \frac{2te^{2t}}{3} + 3\sin 2t + \cos 2t$$

1 c (4 **pts**.) Give a general solution of this equation.

Solution:

$$y = y_h + y_p = c_1e^t + c_2e^{-2t} + \frac{2te^t}{3} + 3\sin 2t + \cos 2t$$

SNB Check: $y'' + y' - 2y = 2e^t - 20\sin 2t$, Exact solution is:
$C_1e^t - \frac{2}{9}e^t + \frac{2}{3}te^t + \cos 2t + 3\sin 2t + C_2e^{-2t}$

2 (40 **pts**. **total**) Consider the Initial Value Problem

$$t^2y'' - 3ty' + 4y = 2t^2 \quad y(1) = 1 \;\; y'(1) = 4 \quad ,y > 0.$$

2a (**5 pts**.) Find a homogeneous solution to this differential equation.

Solution: We first find the homogeneous solution to this Euler equation. The indicial equation is, since $p = -3$ and $q = 4$

$$m^2 + (p-1)m + q = m^2 - 4m + 4 = (m-2)^2$$

Thus $m = 1$ is a repeated root and

$$y_h = c_1t^2 + c_2t^2\ln t$$

2b (**25 pts**.) Find a particular solution of this differential equation.

Solution: To find a particular solution we use the Method of Variations of Parameters with $y_1 = t^2$ and $y_2 = t^2\ln t.$ Then

$$y_p = v_1y_1 + v_2y_2 = t^2v_1 + \left(t^2\ln t\right)v_2$$

and the two equations for v_1' and v_2', namely

$$v_1'y_1 + v_2'y_2 = 0$$
$$v_1'y_1' + v_2'y_2' = \frac{f}{a}$$

become

$$t^2 v_1' + \left(t^2 \ln t\right) v_2' = 0$$
$$2t v_1' + (2t\ln t + t) v_2' = \frac{2t^2}{t^2} = 2$$
$$W[t, t\ln t] = \begin{vmatrix} t^2 & t^2 \ln t \\ 2t & 2t\ln t + t \end{vmatrix} = t^3 \neq 0 \ \text{ since } t > 0$$

Hence

$$v_1' = \frac{\begin{vmatrix} 0 & t^2 \ln t \\ 2 & 2t\ln t + t \end{vmatrix}}{W[t, t\ln t]} = -\frac{2t^2 \ln t}{t^3} = -\frac{2\ln t}{t}$$

$$v_2' = \frac{\begin{vmatrix} t^2 & 0 \\ 2t & 2 \end{vmatrix}}{W[t, t\ln t]} = \frac{2t^2}{t^3} = \frac{2}{t}$$
$$v_1 = \int \left(-\frac{2\ln t}{t}\right) dt = -(\ln t)^2$$
$$v_2 = \int \frac{2}{t} dt = 2\ln t$$

Therefore

$$y_p = y_p = t^2 v_1 + \left(t^2 \ln t\right) v_2 = -t^2 (\ln t)^2 + 2t^2 (\ln t)^2 = t^2 (\ln t)^2$$

2c (10 pts.) Find the solution to this Initial Value Problem given above.

$$y = y_h + y_p = c_1 t^2 + c_2 t^2 \ln t + t^2 (\ln t)^2$$

$$y' = c_1(2t) + c_2\left(2t\ln t + t^2 \cdot \frac{1}{t}\right) + 2t(\ln t)^2 + t^2 (2\ln t)\left(\frac{1}{t}\right)$$
$$= 2tc_1 + c_2(2t\ln t + t) + 2t(\ln t)^2 + 2t\ln t$$

$$y(1) = c_1 = 1$$
$$y'(1) = 2c_1 + c_2 = 4$$

Thus $c_2 = 2$ and

$$y = t^2 + 2t^2 \ln t + t^2 (\ln t)^2$$

3 (30 pts.) Find a general solution of

$$y'' - 2y' = 12x^2$$

Solution: The characteristic equation is

$$p(r) = r^2 - 2r = r(r-2)$$

Thus the roots are $r = 0, 2$ and

$$y_h = c_1 e^{0x} + c_2 e^{2x} = c_1 + c_2 e^{2x}$$

To find a particular solution we note that there is no y term in the DE. Hence we let

$$y_p = Ax + Bx^2 + Cx^3$$

Then

$$y_p' = A + 2Bx + 3Cx^2$$
$$y_p'' = 2B + 6Cx$$

The DE implies

$$2B + 6Cx - 2A - 4Bx - 6Cx^2 = 12x^2$$

Hence

$$C = -2$$
$$2B - 2A = 0$$
$$6C - 4B = 0$$

$B = A = -3$ and

$$y_p = -3x - 3x^2 - 2x^3$$
$$y = y_h + y_p = c_1 + c_2 e^{2x} - 3x - 3x^2 - 2x^3$$

SNB check $y'' - 2y' = 12x^2$, Exact solution is: $C_2 e^{2x} - 3x - 3x^2 - 2x^3 - \frac{1}{2}C_1 - \frac{3}{2}$