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Ma 221 Exam IIIA Solutions 10S

I pledge my honor that I have abided by the Stevens Honor System.

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You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1 ________

#2 ________

#3 ________

#4 ________

Total Score ________

Note: A table of Laplace Transforms is given at the end of the exam.

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1a $\left(13 \textbf{ pts.}\right)$ Use Laplace Transforms to show that the solution $y(t)$ of the initial value problem

$$y'' + 3y' + 2y = 6e^{-t} \quad y(0) = 1 \;\; y'(0) = 2$$

has the Laplace Transform

$$\mathcal{L}\{y\} = \frac{s^2 + 6s + 11}{(s+1)^2(s+2)}$$

Solution: Taking the Laplace Transform of both sides of the DE, we have

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 3s\mathcal{L}\{y\} - 3y(0) + 2\mathcal{L}\{y\} = \frac{6}{s+1}$$
$$(s^2 + 3s + 2)\mathcal{L}\{y\} - s - 2 - 3 = \frac{6}{s+1}$$

Therefore

$$(s^2 + 3s + 2)L\{y\} = \frac{6}{s+1} + s + 5$$
$$\mathcal{L}\{y\} = \frac{s^2 + 6s + 11}{(s+1)^2(s+2)}$$

(**1b** 12 **pts**) Given that

$$\frac{s^2 + 6s + 11}{(s+1)^2(s+2)} = \frac{6}{(s+1)^2} - \frac{2}{s+1} + \frac{3}{s+2}$$

find $y(t)$.

Solution: Below is the derivation of the partial fractions break down that is given above. This was not required in order to get full credit for this problem.

$$\frac{s^2 + 6s + 11}{(s+1)^2(s+2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2}$$

Multiplying by $(s+1)^2$ and setting $s = -1$, yields $B = \frac{1-6+11}{1} = 6$. Multiplying by $s+2$ and setting $s = -2$ yields $C = \frac{4-12+11}{1} = 3$. Thus

$$\frac{s^2 + 6s + 11}{(s+1)^2(s+2)} = \frac{A}{s+1} + \frac{6}{(s+1)^2} + \frac{3}{s+2}$$

Letting $s = 0$ we get $\frac{11}{2} = A + 6 + \frac{3}{2}$, so $A = -2$. Thus

$$\frac{s^2 - 4s + 1}{(s+1)^2(s+2)} = \frac{6}{(s+1)^2} - \frac{2}{s+1} + \frac{3}{s+2}$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left(\frac{s^2 - 4s + 1}{(s+1)^2(s+2)}\right) = \mathcal{L}^{-1}\left(\frac{6}{(s+1)^2}\right) - 2\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + 3\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) \\ &= 6te^{-t} - 2e^{-t} + 3e^{-2t} \end{aligned}$$

SNB check

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$$y'' + 3y' + 2y = 6e^{-t}$$
$$y(0) = 1$$
$$y'(0) = 2$$

, Exact solution is: $\left\{4e^{-t} + 3e^{-2t} + \frac{1}{e^t}(6t-6)\right\}$

2a (15 pts.) Use the definition of the Laplace transform to find $\mathcal{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 0 & 0 \le t < 1 \\ 1 & 1 \le t \le 2 \\ 0 & t \ge 2 \end{cases}$$

Solution:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \int_0^1 0e^{-st}dt + \int_1^2 (1)e^{-st}dt + \int_2^\infty 0e^{-st}dt$$
$$= -\frac{1}{s}e^{-st}\Big|_1^2 = -\frac{1}{s}(e^{-2s} - e^{-s}) = \frac{1}{s}(e^{-s} - e^{-2s})$$

2b $\left(15\ \textbf{pts.}\right)$ Find

$$\mathcal{L}^{-1}\left\{\frac{4s^3 - s^2 + 8s + 4}{s^2(s^2+4)}\right\}$$

Solution: We use partial fractions and

$$\frac{4s^3 - s^2 + 8s + 4}{s^2(s^2+4)} = \frac{A}{s^2} + \frac{B}{s} + \frac{Cs + D}{s^2+4}$$

Putting everything on the right over a common denominator leads to

$$\frac{4s^3 - s^2 + 8s + 4}{s^2(s^2+4)} = \frac{A(s^2+4) + Bs(s^2+4) + Cs^3 + Ds^2}{s^2(s^2+4)}$$

Equating the coefficients of the like powers of s we have

$$B + C = 4$$
$$A + D = -1$$
$$4B = 8$$
$$4A = 4$$

Thus $A = 1, B = 2, C = 2, D = -2$

$$\frac{4s^3 - s^2 + 8s + 4}{s^2(s^2+4)} = \frac{1}{s^2} + \frac{2}{s} + \frac{2s}{s^2+4} - \frac{2}{s^2+4}$$

$$\mathcal{L}^{-1}\left\{\frac{4s^3 - s^2 + 8s + 4}{s^2(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s}\right\} + 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$
$$= t + 2 + 2\cos 2t - \sin 2t$$

3 $\left(25\ \textbf{pts.}\right)$ Find the series solution near $x = 0$ of the equation

$$y'' - x^2 y = 0$$

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Be sure to give the recurrence relation. Indicate the two linearly independent solutions and give the first *six* nonzero terms of the solution.

Solution:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

The DE implies

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

We shift the first series. Let $n - 2 = k + 2$. Then we have that $n = k + 4$ and, since $n = 2$ implies that $k = -2$

$$\sum_{k=-2}^{\infty} a_{k+4}(k+4)(k+3)x^{k+2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Replacing k and n by m and combining the two sums we get

$$a_2(2)(1)x^0 + a_3(3)(2)x + \sum_{m=0}^{\infty}[a_{m+4}(m+4)(m+3) - a_m]x^{m+2} = 0$$

Thus $a_2 = a_3 = 0$ and we have the recurrence relation

$$a_{m+4}(m+4)(m+3) - a_m = 0 \quad \text{for } m = 0, 1, 2, \ldots$$

$$a_{m+4} = \frac{1}{(m+4)(m+3)} a_m \ \text{ for } m = 0, 1, 2, \ldots$$

Hence

$$a_4 = \frac{1}{(4)(3)} a_0$$

$$a_5 = \frac{1}{(5)(4)} a_1$$

$$a_6 = \frac{1}{6(5)} a_2 = 0$$

$$a_7 = 0$$

$$a_8 = \frac{1}{(8)(7)} a_4 = \frac{1}{(8)(7)(4)(3)} a_0$$

$$a_9 = \frac{1}{(9)(8)} a_5 = \frac{1}{(9)(8)(5)(4)} a_1$$

and

$$y(x) = a_0\left[1 + \frac{1}{(4)(3)}x^4 + \frac{1}{(8)(7)(4)(3)}x^8 + \cdots\right] + a_1\left[x + \frac{1}{(5)(4)}x^5 + \frac{1}{(9)(8)(5)(4)}x^9 + \cdots\right]$$

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4 (20 **pts**.) Find the eigenvalues, λ, and eigenfunctions for

$$y'' + (\lambda + 1)y = 0; \quad y'(0) = 0, \quad y'(1) = 0$$

Be sure to consider all values of λ.

Solution: The characteristic equation is

$$r^2 + \lambda + 1 = 0$$

so

$$r = \pm\sqrt{-(\lambda + 1)} = \pm\sqrt{-\lambda - 1}$$

There are 3 cases to consider. $-\lambda - 1 > 0$, $-\lambda - 1 = 0$ and $-\lambda - 1 < 0$.

I. $-\lambda - 1 > 0$ or $-\lambda > 1$ or $\lambda < -1$. Let $-\lambda - 1 = \alpha^2$ where $\alpha \neq 0$. Then $r = \pm\alpha$ and

$$y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

Therefore

$$y'(x) = \alpha(c_1 e^{\alpha x} - c_2 e^{-\alpha x})$$

The boundary condition $y'(0) = 0$ implies

$$c_1 - c_2 = 0 \text{ or } c_1 = c_2$$

Thus

$$y'(x) = \alpha c_1(e^{\alpha x} - e^{-\alpha x})$$

The condition $y'(1) = 0$ implies that $c_1 = 0$. Thus for this case the only solution is $y = 0$. There are no eigenvalues for $\lambda < -1$.

II. $-\lambda - 1 = 0$, that is $\lambda = -1$. Then $r = 0$ and

$$y = c_1 + c_2 x$$

Therefore $y' = c_2$. The boundary conditions imply that $c_2 = 0$, so $y = c_1$, where $c_1 \neq 0$ is a constant is an eigenfunction for the eigenvalue $\lambda = -1$,

III. $-\lambda - 1 < 0$ or $-\lambda < 1$, that is $\lambda > -1$. Let $-\lambda - 1 = -\beta^2$ where $\beta \neq 0$. Thus $\beta^2 = 1 + \lambda$ and $r = \pm\beta i$ and

$$y(x) = c_1 \sin\beta x + c_2 \cos\beta x$$

Hence

$$y'(x) = c_1\beta\cos\beta x - c_2\beta\sin\beta x$$

$$y'(0) = c_1\beta = 0$$

so $c_1 = 0$ and

$$y(x) = c_2\cos\beta x$$

$$y'(x) = -c_2\beta\sin\beta x$$

$$y'(1) = -c_2\beta\sin\beta = 0$$

Thus $\sin\beta = 0$ and $\beta = n\pi,\ \ n = 1,2,3,\ldots.$ Hence

$$\lambda = \beta^2 - 1 = n^2\pi^2 - 1 \quad n = 1,2,3,\ldots.$$

are the eigenvalues and the eigenfunctions are

$$y_n(x) = a_n\cos(n\pi)x \quad n = 1,2,3,\ldots.$$

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Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$		
$\dfrac{t^{n-1}}{(n-1)!}$	$\dfrac{1}{s^n}$	$n \geq 1$	$s > 0$
e^{at}	$\dfrac{1}{s-a}$		$s > a$
$\sin bt$	$\dfrac{b}{s^2 + b^2}$		$s > 0$
$\cos bt$	$\dfrac{s}{s^2 + b^2}$		$s > 0$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		