Ma 221 Exam IIIB Solutions 10S

1a (13 **pts**.) Use Laplace Transforms to show that the solution $y(t)$ of the initial value problem

$$y'' - 3y' + 2y = 6e^t \quad y(0) = 1 \;\; y'(0) = 2$$

has the Laplace Transform

$$\mathcal{L}\{y\} = \frac{s^2 - 2s + 7}{(s-1)^2(s-2)}$$

Solution: Taking the Laplace Transform of both sides of the DE, we have

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 3s\mathcal{L}\{y\} + 3y(0) + 2\mathcal{L}\{y\} = \frac{6}{s-1}$$
$$(s^2 - 3s + 2)\mathcal{L}\{y\} - s - 2 + 3 = \frac{6}{s-1}$$

Therefore

$$(s^2 - 3s + 2)L\{y\} = \frac{6}{s-1} + s - 1$$
$$\mathcal{L}\{y\} = \frac{s^2 - 2s + 7}{(s-1)^2(s-2)}$$

1b (12 **pts**.) Given that

$$\frac{s^2 - 2s + 7}{(s-1)^2(s-2)} = \frac{-6}{(s-1)} + \frac{-6}{(s-1)^2} + \frac{7}{s-2}$$

find $y(t)$.

Solution: Below is the derivation of the partial fractions breakdown given above. This is not required as part of the solution of this problem,

$$\frac{s^2 - 2s + 7}{(s-1)^2(s-2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s-2}$$

Multiplying by $(s-1)^2$ and setting $s = 1$, yields $B = \frac{1-2+7}{-1} = -6$. Multiplying by $s - 2$ and setting $s = 2$ yields $C = \frac{4-4+7}{1} = 7$. Thus

$$\frac{s^2 - 2s + 7}{(s-1)^2(s-2)} = \frac{A}{s-1} + \frac{-6}{(s-1)^2} + \frac{7}{s-2}$$

Letting $s = 0$ we get $\frac{7}{-2} = -A - 6 + \frac{7}{-2}$, so $A = -6$. Thus

$$\frac{s^2 - 2s + 7}{(s-1)^2(s-2)} = \frac{-6}{(s-1)} + \frac{-6}{(s-1)^2} + \frac{7}{s-2}$$

Therefore

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s^2 - 2s + 7}{(s-1)^2(s-2)}\right\} = \mathcal{L}^{-1}\left\{\frac{-6}{(s-1)}\right\} + \mathcal{L}^{-1}\left\{\frac{-6}{(s-1)^2}\right\} + 3\mathcal{L}^{-1}\left\{\frac{7}{s-2}\right\}$$
$$= -6e^t - 6te^t + 7e^{2t}$$

Solutions - Exam IIIB

SNB check

$$\begin{aligned} y'' - 3y' + 2y &= 6e^t \\ y(0) &= 1 \\ y'(0) &= 2 \end{aligned}$$

, Exact solution is: $\{7e^{2t} - 6e^t - 6te^t\}$

2a (15 pts.) Use the definition of the Laplace transform to find $\mathcal{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 0 & 0 \le t < 1 \\ 2 & 1 \le t \le 3 \\ 0 & t \ge 2 \end{cases}$$

Solution:

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \int_0^\infty f(t)e^{-st}dt = \int_0^1 0e^{-st}dt + \int_1^3 (2)e^{-st}dt + \int_3^\infty 0e^{-st}dt \\ &= -\frac{2}{s}e^{-st}\Big|_1^3 = -\frac{2}{s}(e^{-3s} - e^{-s}) = \frac{2}{s}(e^{-s} - e^{-3s}) \end{aligned}$$

2b (15 **pts.**) Find

$$\mathcal{L}^{-1}\left\{\frac{4s^3 - s^2 + 4s + 8}{s^2(s^2+4)}\right\}$$

Solution: Method 1 without using complex variables.

$$\frac{4s^3 - s^2 + 4s + 8}{s^2(s^2+4)} = \frac{A}{s^2} + \frac{B}{s} + \frac{Cs + D}{s^2 + 4}$$

: $\frac{1}{s^2(s^2+4)}(4A + As^2 + Bs^3 + Cs^3 + s^2D + 4Bs)$

Putting everything on the right over a common denominator leads to

$$\frac{4s^3 - s^2 + 4s + 8}{s^2(s^2+4)} = \frac{A(s^2+4) + Bs(s^2+4) + Cs^3 + Ds^2}{s^2(s^2+4)}$$

Equating the coefficients of the like powers of s we have

$$\begin{aligned} B + C &= 4 \\ A + D &= -1 \\ 4B &= 4 \\ 4A &= 8 \end{aligned}$$

Thus $A = 2, B = 1, C = 3, D = -3$

$$\frac{4s^3 - s^2 + 8s + 4}{s^2(s^2+4)} = \frac{2}{s^2} + \frac{1}{s} + \frac{3s}{s^2+4} - \frac{3}{s^2+4}$$

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{4s^3 - s^2 + 4s + 8}{s^2(s^2+4)}\right\} &= \mathcal{L}^{-1}\left\{\frac{2}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\ &= 2t + 1 + 3\cos 2t - \frac{3}{2}\sin 2t \end{aligned}$$

3 (25 **pts.**) Find the series solution near $x = 0$ of the equation

$$y'' + x^2 y = 0$$

Be sure to give the recurrence relation. Indicate the two linearly independent solutions and give the first *six* nonzero terms of the solution.

Solution:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

The DE implies

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

We shift the first series. Let $n - 2 = k + 2.$ Then we have that $n = k + 4$ and, since $n = 2$ implies that $k = -2$

$$\sum_{k=-2}^{\infty} a_{k+4}(k+4)(k+3)x^{k+2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Replacing k and n by m and combining the two sums we get

$$a_2(2)(1)x^0 + a_3(3)(2)x + \sum_{m=0}^{\infty} [a_{m+4}(m+4)(m+3) + a_m]x^{m+2} = 0$$

Thus $a_2 = a_3 = 0$ and we have the recurrence relation

$$a_{m+4}(m+4)(m+3) + a_m = 0 \quad \text{for } m = 0, 1, 2, \ldots$$

$$a_{m+4} = \frac{-1}{(m+4)(m+3)} a_m \ \text{ for } m = 0, 1, 2, \ldots$$

Hence

$$a_4 = \frac{-1}{(4)(3)} a_0$$

$$a_5 = \frac{-1}{(5)(4)} a_1$$

$$a_6 = \frac{-1}{6(5)} a_2 = 0$$

$$a_7 = 0$$

$$a_8 = \frac{-1}{(8)(7)} a_4 = \frac{1}{(8)(7)(4)(3)} a_0$$

$$a_9 = \frac{-1}{(9)(8)} a_5 = \frac{1}{(9)(8)(5)(4)} a_1$$

and

$$y(x) = a_0\left[1 + \frac{-1}{(4)(3)}x^4 + \frac{1}{(8)(7)(4)(3)}x^8 + \cdots\right] + a_1\left[x + \frac{-1}{(5)(4)}x^5 + \frac{1}{(9)(8)(5)(4)}x^9 + \cdots\right]$$

4 $\left(20\ \textbf{pts.}\right)$ Find the eigenvalues, λ, and eigenfunctions for

$$y'' + (\lambda - 1)y = 0; \quad y'(0) = 0, \quad y'(1) = 0$$

Be sure to consider all values of λ.

Solution: The characteristic equation is

$$r^2 + \lambda - 1 = 0$$

so

$$r = \pm\sqrt{-(\lambda - 1)} = \pm\sqrt{-\lambda + 1}$$

There are 3 cases to consider. $-\lambda + 1 > 0$, $-\lambda + 1 = 0$ and $-\lambda + 1 < 0$.

I. $-\lambda + 1 > 0$ or $-\lambda > -1$ or $\lambda < 1$. Let $-\lambda + 1 = \alpha^2$ where $\alpha \neq 0$. Then $r = \pm\alpha$ and

$$y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

Therefore

$$y'(x) = \alpha(c_1 e^{\alpha x} - c_2 e^{-\alpha x})$$

The boundary condition $y'(0) = 0$ implies

$$c_1 - c_2 = 0 \text{ or } c_1 = c_2$$

Thus

$$y'(x) = \alpha c_1 (e^{\alpha x} - e^{-\alpha x})$$

The condition $y'(1) = 0$ implies that $c_1 = 0$. Thus for this case the only solution is $y = 0$. There are no eigenvalues for $\lambda < -1$.

II. $-\lambda + 1 = 0$, that is $\lambda = 1$. Then $r = 0$ and

$$y = c_1 + c_2 x$$

Therefore $y' = c_2$. The boundary conditions imply that $c_2 = 0$, so $y = c_1$, where $c_1 \neq 0$ is a constant is an eigenfunction for the eigenvalue $\lambda = 1$,

III. $-\lambda + 1 < 0$ or $-\lambda < -1$, that is $\lambda > 1$. Let $-\lambda + 1 = -\beta^2$ where $\beta \neq 0$. Thus $\beta^2 = \lambda - 1$ and $r = \pm\beta i$ and

$$y(x) = c_1 \sin\beta x + c_2 \cos\beta x$$

Hence

$$y'(x) = c_1 \beta \cos\beta x - c_2 \beta \sin\beta x$$

$$y'(0) = c_1 \beta = 0$$

so $c_1 = 0$ and

$$y(x) = c_2 \cos\beta x$$
$$y'(x) = -c_2 \beta \sin\beta x$$

$$y'(1) = -c_2 \beta \sin\beta = 0$$

Thus $\sin\beta = 0$ and $\beta = n\pi$, $n = 1, 2, 3, \ldots$. Hence

$$\lambda = 1 + \beta^2 = 1 + n^2\pi^2 \quad n = 1, 2, 3, \ldots.$$

are the eigenvalues and the eigenfunctions are

$$y_n(x) = a_n \cos(n\pi x) \quad n = 1, 2, 3, \ldots.$$

Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$		
$\dfrac{t^{n-1}}{(n-1)!}$	$\dfrac{1}{s^{n}}$	$n \geq 1$	$s > 0$
e^{at}	$\dfrac{1}{s-a}$		$s > a$
$\sin bt$	$\dfrac{b}{s^2 + b^2}$		$s > 0$
$\cos bt$	$\dfrac{s}{s^2 + b^2}$		$s > 0$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		