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Ma 221 Exam IA Solutions
12S

Solve:

1 [25 **pts**.]

$$(e^t x+1)dt+(e^t-1)dx = 0 \quad x(1) = 1$$

Solution:

$$M = e^t x+1 \quad \text{and} \quad N = e^t-1$$

Since

$$M_x = e^t = N_t$$

the equation is exact. Thus there exists $f(t,x)$ such that

$$f_t = M \quad \text{and} \quad f_x = N$$

From the first equation we get by integrating with respect to t while holding x fixed

$$f = e^t x+t+g(x)$$

But

$$f_x = e^t+g'(x) = N = e^t-1$$

Hence

$$g'(x) = -1$$

and $g(x) = -x+c$ so

$$f = e^t x+t-x+c$$

and the solution is given by

$$e^t x+t-x = k$$

The initial condition implies

$$e+1-1 = k$$

so the solution is given by

$$e^t x+t-x = e$$

or

$$x = \frac{e-t}{e^t-1}$$

2 [25 **pts**.]

$$\frac{dy}{dx}+4y-e^{-x} = 0 \quad y(0) = \frac{4}{3}$$

Solution: This equation is first order linear, and we rewrite it as

$$\frac{dy}{dx}+4y = e^{-x}$$

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Since $P(x) = 4$ the integrating factor is

$$e^{\int 4dx} = e^{4x}$$

Multiplying both sides of the DE by this we get

$$e^{4x}y' + 4e^{4x}y = e^{3x}$$

or

$$\frac{d\left(e^{4x}y\right)}{dx} = e^{3x}$$

Integrating we have

$$e^{4x}y = \frac{1}{3}e^{3x} + c$$

or

$$y = \frac{1}{3}e^{-x} + ce^{-4x}$$

The initial condition implies

$$\frac{4}{3} = \frac{1}{3} + c$$

so $c = 1$ and

$$y = \frac{1}{3}e^{-x} + e^{-4x}$$

3 [25 **points**]

$$\left(x + xy^2\right)dx + e^{x^2}ydy = 0$$

Solution: This equation is separable, since it can be rewritten as

$$x\left(1 + y^2\right)dx + e^{x^2}ydy = 0$$

or

$$xe^{-x^2}dx = -\frac{y}{1 + y^2}dy$$

Integrating we have

$$-\frac{1}{2}e^{-x^2} = -\frac{1}{2}\ln\left(1 + y^2\right) + c$$

or

$$\ln\left(1 + y^2\right) - e^{-x^2} = k$$

4 [25 **pts**.]

$$y' + \frac{4}{x}y = x^3y^2 \quad y(2) = -1$$

Solution: This is a Bernoulli equation. Multiplying both side of the equation by y^{-2} leads to

$$y^{-2}y' + \frac{4}{x}y^{-1} = x^3$$

Let $z = y^{-1}$. Then $z' = -y^{-2}y'$ and we may rewrite the DE as

$$-z' + \frac{4}{x}z = x^3$$

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or

$$z' - \frac{4}{x} z = -x^3$$

which is a first order linear DE in z. The integrating factor for this equation is

$$e^{-\int \frac{4}{x} dx} = e^{-4 \ln x} = x^{-4}$$

Multiplying the DE for z by x^{-4} leads to

$$x^{-4} z' - 4x^{-5} z = -x^{-1}$$

or

$$\frac{d\left(x^{-4} z\right)}{dx} = -x^{-1}$$

or

$$x^{-4} z = -\ln x + c$$

Thus

$$z = -x^4 \ln x + cx^4$$

Since $z = y^{-1}$ we have

$$\frac{1}{y} = -x^4 \ln x + cx^4$$

The initial condition implies

$$-1 = -2^4 \ln 2 + 2^4 c$$

or

$$c = -\frac{1}{16} + \ln 2$$

Finally we have

$$\frac{1}{y} = -x^4 \ln x + \left(-\frac{1}{16} + \ln 2\right) x^4$$