

**Ma 221
12S****Exam II A Solutions****1. (40 pts. total)** Consider the Initial Value Problem

$$y'' + 2y' + y = t^2 + 1 - e^t \quad y(0) = 0 \quad y'(0) = 2$$

1 a (6 pts.) Find the homogeneous solution of this equation.

Solution: The characteristic polynomial is

$$p(r) = r^2 + 2r + 1 = (r + 1)^2$$

Thus $r = -1$ is a repeated root and

$$y_h = c_1 e^{-t} + c_2 t e^{-t}$$

1 b (20 pts.) Find a particular solution of this equation.Solution: We first find a particular solution for $-e^t$. Since e^t is not a homogeneous solution, then we use the formula

$$y_{p1} = \frac{ke^{\alpha t}}{p(\alpha)}$$

Here $\alpha = 1$ and $k = -1$ so since $p(1) = 4$

$$y_{p1} = -\frac{e^t}{4}$$

To find a particular solution for $t^2 + 1$ we let

$$y_{p2} = A_2 t^2 + A_1 t + A_0$$

$$y'_{p2} = 2A_2 t + A_1$$

$$y''_{p2} = 2A_2$$

Plugging into the DE we have

$$2A_2 + 4A_2 t + 2A_1 + A_2 t^2 + A_1 t + A_0 = t^2 + 1$$

Thus $A_2 = 1$, $4A_2 + A_1 = 0$, $2A_2 + 2A_1 + A_0 = 1$. Then $A_1 = -4A_2 = -4$, and

$$A_0 = 1 - 2A_2 - 2A_1 = 1 - 2 + 8 = 7$$

Thus

$$y_{p2} = t^2 - 4t + 7$$

Finally

$$y_p = y_{p1} + y_{p2} = -\frac{e^t}{4} + t^2 - 4t + 7$$

1 c (4 pts.) Give a general solution of this equation.

$$y = y_h + y_p = c_1 e^{-t} + c_2 t e^{-t} - \frac{e^t}{4} + t^2 - 4t + 7$$

1d (10 pts.) Find the solution to this Initial Value Problem

$$y'' + 2y' + y = t^2 + 1 - e^t \quad y(0) = 0 \quad y'(0) = 2$$

Solution:

$$y(0) = c_1 - \frac{1}{4} + 7 = 0 \Rightarrow c_1 = -\frac{27}{4}$$

$$y'(t) = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t} - \frac{e^t}{4} + 2t - 4$$

$$y'(0) = -c_1 + c_2 - \frac{1}{4} - 4 = 2$$

$$c_2 = c_1 + \frac{17}{4} + 2 = -\frac{27}{4} + \frac{17}{4} + 2 = -\frac{1}{2}$$

$$y = -\frac{27}{4}e^{-t} - \frac{1}{2}te^{-t} - \frac{e^t}{4} + t^2 - 4t + 7$$

$$y'' + 2y' + y = t^2 + 1 - e^t$$

$$y(0) = 0$$

$$y'(0) = 2$$

2 (20 pts.) Find a general solution of

$$t^2 y'' + 3ty' + 5y = 0$$

Solution: This is a Cauchy-Euler equation with $p = 3$ and $q = 5$. Thus if t^m is a solution then the equation for m is

$$m^2 + (p-1)m + q = 0$$

or

$$m^2 + 2m + 5 = 0$$

$$m = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

Then $a = -1$ and $b = 2$ and the formula

$$y_h = t^a [A \cos(b \ln t) + B \sin(b \ln t)].$$

becomes for this problem

$$y_h = t^{-1} [A \cos(2 \ln t) + B \sin(2 \ln t)]$$

3 (20 pts.) Find a general solution of the differential equation

$$y'' - 6y' + 9y = t^{-3}e^{3t}$$

Solution: The characteristic polynomial is

$$p(r) = r^2 - 6r + 9 = (r-3)^2$$

Thus $r = 3$ is a repeated root and

$$y_h = c_1 e^{3t} + c_2 t e^{3t}$$

We will use the Method of Variation of Parameters with $y_1 = e^{3t}$ and $y_2 = t e^{3t}$ so that

$$y_p = v_1 e^{3t} + v_2 t e^{3t}$$

The two equations for v_1' and v_2' , namely

$$v_1' y_1 + v_2' y_2 = 0$$

$$v_1' y_1' + v_2' y_2' = \frac{f}{a}$$

become for this problem

$$v_1' e^{3t} + v_2' t e^{3t} = 0$$

$$3v_1' e^{3t} + v_2' (e^{3t} + 3t e^{3t}) = t^{-3} e^{3t}$$

We may cancel e^{3t} in both equations to get

$$v_1' + v_2' t = 0$$

$$3v_1' + v_2' (1 + 3t) = t^{-3}$$

$$W[e^{3t}, t e^{3t}] = \begin{vmatrix} 1 & t \\ 3 & 1 + 3t \end{vmatrix} = 1, \text{ so}$$

$$v_1' = \frac{\begin{vmatrix} 0 & t \\ t^{-3} & 1 + 3t \end{vmatrix}}{1} = -t^{-2}$$

$$v_2' = \frac{\begin{vmatrix} 1 & 0 \\ 3 & t^{-3} \end{vmatrix}}{1} = t^{-3}$$

Therefore

$$v_1 = \frac{1}{t} \quad \text{and} \quad v_2 = -\frac{1}{2} t^{-2}$$

$$y_p = v_1 e^{3t} + v_2 t e^{3t} = \frac{1}{t} e^{3t} - \frac{1}{2} t^{-1} e^{3t} = \frac{e^{3t}}{2t}$$

Finally

$$y = y_h + y_p = c_1 e^{3t} + c_2 t e^{3t} + \frac{e^{3t}}{2t}$$

4 (15 pts.) Write down a second order homogeneous linear differential equation with real constant coefficients of the form

$$y'' + by' + cy = 0$$

whose solutions are

$$\frac{1}{2} e^{-2x} \cos 3x \quad \text{and} \quad \frac{3e^{-2x}}{4} \sin 3x.$$

Solution: These solutions come from complex roots $\alpha \pm i\beta$ of the characteristic equation

$$p(r) = r^2 + br + c = 0$$

$$\Rightarrow \alpha = -2 \quad \beta = 3 \text{ so that } r_1 = -2 + 3i \text{ and } r_2 = -2 - 3i.$$

\Rightarrow

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$$\begin{aligned} p(r) &= [r - (-2 + 3i)][r - (-2 - 3i)] \\ &= [r + 2 - 3i][r + 2 + 3i] \\ &= r^2 + (2 + 3i)r + (2 - 3i)r + 4 + 9 \\ &= r^2 + 4r + 13 \end{aligned}$$

$$(\text{Check: } r = \frac{-4 \pm \sqrt{16 - 4(1)(13)}}{2} = \frac{-4 \pm \sqrt{36}i}{2} = -2 \pm 3i)$$

\Rightarrow equation is

$$y'' + 4y' + 13y = 0$$

Alternative Solution: Since $a = 1$ the roots of the characteristic polynomial are given by

$$r = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Thus

$$-\frac{b}{2} = \alpha = -2$$

so $b = 4$.

$$\frac{\sqrt{b^2 - 4c}}{2} = 3i$$

so

$$b^2 - 4c = -36$$

Hence $16 - 4c = -36$ and $c = 13$

Table of Integrals

$$\int \ln t dt = t(\ln t - 1) + C$$

$$\int (\ln t)^2 dt = t(\ln^2 t - 2 \ln t + 2) + C$$

$$\int \frac{\ln t}{t} dt = \frac{1}{2} \ln^2 t + C$$

$$\int \frac{(\ln t)^2}{t} dt + C = \frac{1}{3} \ln^3 t + C$$