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Ma 221 **Exam II B Solutions**
12S

1. (40 **pts**. **total**) Consider the Initial Value Problem

$$y'' - 2y' + y = t^2 + 1 - e^{-t} \;\; y(0) = 0 \;\; y'(0) = 2$$

1 a (6 **pts**.) Find the homogeneous solution of this equation.

Solution: The characteristic polynomial is

$$p(r) = r^2 - 2r + 1 = (r-1)^2$$

Thus $r = 1$ is a repeated root and

$$y_h = c_1 e^t + c_2 t e^t$$

1 b (20 **pts**.) Find a particular solution of this equation.

Solution: We first find a particular solution for $-e^{-t}$. Since e^{-t} is not a homogeneous solution, then we use the formula

$$y_{p_1} = \frac{k e^{\alpha t}}{p(\alpha)}$$

Here $\alpha = -1$ and $k = -1$ so since $p(1) = 4$

$$y_{p_1} = -\frac{e^{-t}}{4}$$

To find a particular solution for $t^2 + 1$ we let

$$y_{p_2} = A_2 t^2 + A_1 t + A_0$$
$$y'_{p_2} = 2A_2 t + A_1$$
$$y''_{p_2} = 2A_2$$

Plugging into the DE we have

$$2A_2 - 4A_2 t - 2A_1 + A_2 t^2 + A_1 t + A_0 = t^2 + 1$$

Thus $A_2 = 1$, $-4A_2 + A_1 = 0$, $2A_2 - 2A_1 + A_0 = 1$. Then $A_1 = 4A_2 = 4$, and

$$A_0 = 1 - 2A_2 + 2A_1 = 1 - 2 + 8 = 7$$

Thus

$$y_{p_2} = t^2 + 4t + 7$$

Finally

$$y_p = y_{p_1} + y_{p_2} = -\frac{e^{-t}}{4} + t^2 + 4t + 7$$

1 c (4 **pts**.) Give a general solution of this equation.

$$y = y_h + y_p = c_1 e^t + c_2 t e^t - \frac{e^{-t}}{4} + t^2 + 4t + 7$$

1d (10 **pts**.) Find the solution to this Initial Value Problem

$$y'' - 2y' + y = t^2 + 1 - e^{-t} \;\; y(0) = 0 \;\; y'(0) = 2$$

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Solution:

$$y(0) = c_1 - \frac{1}{4} + 7 = 0 \quad \Rightarrow c_1 = -\frac{27}{4}$$

$$y'(t) = c_1 e^t + c_2 e^t + c_2 t e^t - \frac{e^t}{4} + 2t + 4$$

$$y'(0) = c_1 + c_2 - \frac{1}{4} + 4 = 2$$

$$c_2 = -c_1 - \frac{15}{4} + 2 = \frac{27}{4} - \frac{15}{4} + 2 = 5$$

$$y = -\frac{27}{4} e^t + 5te^t - \frac{e^t}{4} + t^2 + 4t + 7$$

2 (20 **pts**.) Find a general solution of

$$t^2 y'' - ty' + 5y = 0$$

Solution: This is a Cauchy-Euler equation with $p = -1$ and $q = 5.$ Thus if t^r is a solution then the equation for m is

$$r^2 + (p-1)r + q = 0$$

or

$$\begin{aligned} r^2 - 2r + 5 &= 0 \\ \left(r^2 - 2r + 1\right) + 4 &= 0 \\ (r-1)^2 &= -4 \\ r - 1 &= \pm 2i \\ r &= 1 \pm 2i \end{aligned}$$

Then $a = 1$ and $b = 2$ and the formula

$$y_h = t^a[c_1 \cos(b \ln t) + c_2 \sin(b \ln t)].$$

becomes for this problem

$$y_h = t[c_1 \cos(2 \ln t) + c_2 \sin(2 \ln t)]$$

Alternatively, a complex solustion is

$$\begin{aligned} y &= t^{1+2i} = t \bullet t^{2i} \\ &= t \bullet \left(e^{\ln t}\right)^{2i} = t \bullet e^{i 2 \ln t} \\ &= t[\cos(2 \ln t) + i \sin(2 \ln t)] \end{aligned}$$

and the resullt stated follows by taking the real and imaginary parts.

3 (**20 pts**.) Find a general solution of the differential equation

$$y'' - 6y' + 9y = t^{-3} e^{-3t}$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 6r + 9 = (r+3)^2$$

Thus $r = -3$ is a repeated root and

$$y_h = c_1 e^{-3t} + c_2 t e^{-3t}$$

We will use the Method of Variation of Parameters with $y_1 = e^{-3t}$ and $y_2 = te^{-3t}$ so that

$$y_p = v_1 e^{-3t} + v_2 t e^{-3t}$$

The two equations for v_1' and v_2', namely

$$v_1' y_1 + v_2' y_2 = 0$$
$$v_1' y_1' + v_2' y_2' = \frac{f}{a}$$

become for this problem

$$v_1' e^{-3t} + v_2' t e^{-3t} = 0$$
$$-3v_1' e^{-3t} + v_2'\left(e^{-3t} - 3te^{-3t}\right) = t^{-3} e^{-3t}$$

We may cancel e^{-3t} in both equations to get

$$v_1' + v_2' t = 0$$
$$-3v_1' + v_2'(1 - 3t) = t^{-3}$$

Multiply the first equation by 3 and add the second to obtain

$$v_2' = t^{-3}$$

Substitute this in the first equation to obtain

$$v_1' = -tv_2' = -t^{-2}$$

Therefore

$$v_1 = \frac{1}{t} \quad \text{and} \quad v_2 = -\frac{1}{2} t^{-2}$$

$$y_p = v_1 e^{-3t} + v_2 t e^{-3t} = \frac{1}{t} e^{-3t} - \frac{1}{2} t^{-1} e^{-3t} = \frac{e^{-3t}}{2t}$$

Finally

$$y = y_h + y_p = c_1 e^{-3t} + c_2 t e^{-3t} + \frac{e^{-3t}}{2t}$$

4 (15 **pts**.) Write down a second order homogeneous linear differential equation with real constant coefficients of the form

$$y'' + by' + cy = 0$$

whose solutions are

$$\frac{1}{2} e^{-4x} \cos 3x \quad \text{and} \quad \frac{3e^{-4x}}{4} \sin 3x.$$

Solution: These solutions come from complex roots $\alpha \pm i\beta$ of the characteristic equation

$$p(r) = r^2 + br + d = 0$$

$\Rightarrow \alpha = -4 \quad \beta = 3$ so that $r_1 = -4 + 3i$ and $r_2 = -4 - 3i$.

$\Rightarrow$

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$$\begin{aligned} p(r) &= [r - (-4 + 3i)][r - (-4 - 3i)] \\ &= [(r + 4) - 3i][(r + 4) + 3i] \\ &= (r + 4)^2 + 9 \\ &= r^2 + 8r + 25 \end{aligned}$$

(Check: $r = \dfrac{-8 \pm \sqrt{64 - 4(1)(25)}}{2} = \dfrac{-8 \pm \sqrt{36}\, i}{2} = -4 \pm 3i$)

$\Rightarrow$ equation is

$$y'' + 8y' + 25y = 0$$

Alternative Solution: Since $a = 1$ the roots of the characteristic poynomial are given by

$$r = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Thus

$$-\frac{b}{2} = \alpha = -4$$

so $b = 8$.

$$\frac{\sqrt{b^2 - 4c}}{2} = 3i$$

so

$$b^2 - 4c = -36$$

Hence $64 - 4\,c = -36$ and $c = 25$

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Table of Integrals

$$\int \ln t\,dt = t(\ln t - 1) + C$$

$$\int (\ln t)^2 dt = t\left(\ln^2 t - 2\ln t + 2\right) + C$$

$$\int \frac{\ln t}{t} dt = \frac{1}{2}\ln^2 t + C$$

$$\int \frac{(\ln t)^2}{t} dt + C = \frac{1}{3}\ln^3 t + C$$