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Ma 221 Exam IIIA Solutions 12S

I pledge my honor that I have abided by the Stevens Honor System. ____

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1 ________

#2 ________

#3 ________

#4 ________

Total Score ________

Note: A table of Laplace Transforms is given at the end of the exam.

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1 (25 **pts.**) Use Laplace Transforms to solve the initial value problem

$$y'' + 6y' + 5y = 12e^t \quad y(0) = -1 \;\; y'(0) = 7$$

Solution: Taking Laplace transforms of the DE and letting $\mathcal{L}\{y\} = Y$ we have

$$s^2Y - sy(0) - y'(0) + 6sY - 6y(0) + 5Y = \frac{12}{s-1}$$

or

$$(s^2 + 6s + 5)Y = \frac{12}{s-1} - s + 7 - 6$$

Thus

$$\begin{aligned} Y &= \left(\frac{1}{(s+1)(s+5)}\right)\left(\frac{12}{s-1} - (s-1)\right) \\ &= \frac{-s^2 + 2s + 11}{(s+1)(s+5)(s-1)} \end{aligned}$$

To invert this we express the last fraction in partial fractions.

$$\frac{-s^2 + 2s + 11}{(s+1)(s+5)(s-1)} = \frac{A}{s+1} + \frac{B}{s+5} + \frac{C}{s-1}$$

Multiplying by $s + 1$ and setting $s = -1$ we get

$$\frac{-(-1)^2 - 2 + 11}{4(-2)} = A$$

so $A = -1$. Similarly

$$\frac{-(-5)^2 - 10 + 11}{(-4)(-6)} = B$$

so $B = -1$ and

$$\frac{-1 + 2 + 11}{2(6)} = C$$

and $C = 1$. Thus

$$Y = -\frac{1}{s+1} - \frac{1}{s+5} + \frac{1}{s-1}$$

so

$$y(t) = -e^{-t} - e^{-5t} + e^t$$

2 (20 **pts.**) Find

$$\mathcal{L}^{-1}\left\{\frac{-5s - 36}{(s+2)(s^2+9)}\right\}$$

Solution:

$$\frac{-5s - 36}{(s+2)(s^2+9)} = \frac{A}{s+2} + \frac{Bs + C}{s^2 + 9}$$

Multiplying both sides by $s + 2$ and letting $s = -2$ yields

$$\frac{10 - 36}{13} = -2 = A$$

And

$$\frac{-5s - 36}{(s+2)(s^2+9)} = \frac{-2}{s+2} + \frac{Bs + C}{s^2 + 9}$$

Letting $s = 0$ yields

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$$\frac{-36}{18} = -2 = -1 + \frac{C}{9}$$

Thus $C = -9$ and we have

$$\frac{-5s-36}{(s+2)(s^2+9)} = \frac{-2}{s+2} + \frac{Bs-9}{s^2+9}$$

Letting $s = 1$ yields

$$\frac{-5-36}{3(10)} = \frac{-2}{3} + \frac{B-9}{10}$$

Multiplying by 30 yields

$$-41 = -20 + 3B - 27$$

or $3B = 6$ so $B = 2$.
Thus

$$\frac{-5s-36}{(s+2)(s^2+9)} = \frac{-2}{s+2} + \frac{2s-9}{s^2+9}$$

$$\mathcal{L}^{-1}\left\{\frac{-5s-36}{(s+2)(s^2+9)}\right\} = -2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - 3\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$
$$= -2e^{-2t} + 2\cos 3t - 3\sin 3t$$

3 (30 **pts**.) Find the series solution near $x = 0$ of the equation

$$(x^2+1)y'' + y = 0$$

Be sure to give the recurrence relation. Indicate the two linearly independent solutions and give the first *five* nonzero terms of the solution.

Solution:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

Substituting into the DE leads to

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Combining the first and third series we have

$$a_0 + a_1 x + \sum_{n=2}^{\infty} a_n[n(n-1)+1]x^n + \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} = 0$$

Let $n-2 = k$ or $n = k+2$ in the second series. Then we have

$$a_0 + a_1 x + \sum_{n=2}^{\infty} a_n[n(n-1)+1]x^n + \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k = 0$$

Replacing n and k by m we have

$$a_0 + a_1x + 2a_2 + 3(2)a_3x + \sum_{m=2}^{\infty}\{a_m[m(m-1)+1] + a_{m+2}(m+2)(m+1)\}x^m = 0$$

Thus

$$a_2 = -\frac{1}{2}a_0$$

$$a_3 = -\frac{1}{6}a_1$$

and the recurrence relation is

$$a_m(m^2 - m + 1) + a_{m+2}(m+2)(m+1) = 0 \quad m = 2,3,\ldots$$

or

$$a_{m+2} = -\frac{m^2 - m + 1}{(m+2)(m+1)}a_m \quad m = 2,3,\ldots$$

Hence

$$a_4 = -\frac{3}{4(3)}a_2 = -\frac{1}{4}a_2 = \frac{1}{8}a_0$$

$$a_5 = -\frac{7}{5(4)}a_3 = \frac{7}{6(5)(4)}a_1$$

Thus

$$y = a_0\left(1 - \frac{x^2}{2} + \frac{1}{8}x^4 + \cdots\right) + a_1\left(x - \frac{1}{6}x^3 + \frac{7}{6(5)(4)}x^5 + \cdots\right)$$

4 (25 **pts**.) Find the eigenvalues, λ, and eigenfunctions for

$$y'' + \lambda y = 0; \quad y(0) = 0, \quad y'(1) = 0$$

Be sure to consider all values of λ.
Solution: We must consider three cases; $\lambda < 0$, $\lambda = 0$, , and $\lambda > 0$.

I. $\lambda < 0$. Let $\lambda = -\alpha^2$ where $\alpha \neq 0$. Then the differential equation becomes

$$y'' - \alpha^2 y = 0$$

and has the general solution

$$y(x) = c_1e^{\alpha x} + c_2e^{-\alpha x}.$$

$y(0) = 0 \Rightarrow c_1 = -c_2$ so

$$y = c_1(e^{\alpha x} - e^{-\alpha x})$$

Thus

$$y'(x) = c_1\alpha(e^{\alpha x} + e^{-\alpha x})$$

$y'(1) = 0 \Rightarrow$

$$c_1\alpha(e^{\alpha} + e^{-\alpha}) = 0$$

Thus $c_1 = 0$ and hence $c_2 = 0$ and the only solution for this case is $y = 0$.

II. $\lambda = 0$. The solution is $y = c_1x + c_2$. The BCs imply $c_1 = c_2 = 0$. Again the only solution is $y = 0$.

III. $\lambda > 0$. Let $\lambda = \beta^2$ where $\beta \neq 0$. The DE becomes

$$y'' + \beta^2 y = 0$$

and has the general solution

$$y = c_1 \sin \beta x + c_2 \cos \beta x.$$

The BCs imply

$$y(0) = c_2 = 0$$

so

$$y(x) = c_1 \sin \beta x$$

$$y'(x) = c_1 \beta \cos \beta x$$

$$y'(1) = c_1 \beta \cos \beta = 0$$

Hence

$$\beta = \left(\frac{2n+1}{2}\right)\pi \quad n = 0, 1, 2, \ldots$$

Hence the eigenvalues are

$$\lambda = \beta^2 = \left(\frac{2n+1}{2}\right)^2 \pi^2 \quad n = 0, 1, 2, \ldots$$

and the eigenfunctions are

$$y_n(x) = a_n \sin\left(\frac{2n+1}{2}\right)\pi x \quad n = 0, 1, 2, \ldots$$

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Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$		
$\dfrac{t^{n-1}}{(n-1)!}$	$\dfrac{1}{s^n}$	$n \geq 1$	$s > 0$
e^{at}	$\dfrac{1}{s-a}$		$s > a$
$\sin bt$	$\dfrac{b}{s^2 + b^2}$		$s > 0$
$\cos bt$	$\dfrac{s}{s^2 + b^2}$		$s > 0$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		