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Ma 221 Exam IIIB Solutions 12S

1 (25 **pts**.) Use Laplace Transforms to solve the initial value problem

$$y'' - 6y' + 5y = 12e^t \quad y(0) = 1 \;\; y'(0) = 3$$

Solution: Taking Laplace transforms of the DE and letting $\mathcal{L}\{y\} = Y$ we have

$$s^2Y - sy(0) - y'(0) - 6sY + 6y(0) + 5Y = \frac{12}{s-1}$$

or

$$(s^2 - 6s + 5)Y = \frac{12}{s-1} + s + 3 - 6$$

Thus

$$\begin{aligned} Y &= \left(\frac{1}{(s-1)(s-5)}\right)\left(\frac{12}{s-1} + (s-3)\right) \\ &= \frac{12 + (s-1)(s-3)}{(s-1)^2(s-5)} \end{aligned}$$

To invert this we express the last fraction in partial fractions.

$$\frac{12 + (s-1)(s-3)}{(s-1)^2(s-5)} = \frac{A}{s-1} + \frac{B}{s-5} + \frac{C}{(s-1)^2}$$

Multiplying by the common denominator gives

$$12 + (s-1)(s-3) = A(s-5)(s-1) + B(s-1)^2 + C(s-5)$$

Setting $s = 1$ we get

$$12 = -4C$$

so $C = -3$. Similarly with $s = 5$,

$$12 + (4)2 = 16B$$
$$16B = 20$$

so $B = \frac{5}{4}$. Thus we have

$$12 + (s-1)(s-3) = A(s-5)(s-1) + \frac{5}{4}(s-1)^2 - 3(s-5)$$

and from setting $s = 0$

$$12 + (-1)(-3) = A(-5)(-1) + \frac{5}{4} + 15$$

or

$$15 = 5A + \frac{5}{4} + 15$$

so

$$5A = -\frac{5}{4}$$

and $A = -\frac{1}{4}$. Thus

$$Y = \frac{-\frac{1}{4}}{s-1} + \frac{\frac{5}{4}}{s-5} + \frac{-3}{(s-1)^2}$$

so

$$y(t) = -\frac{1}{4}e^{-t} + \frac{5}{4}e^{5t} - 3te^{t}$$

Exact solution is: $\left\{\frac{5}{4}e^{5t} - \frac{1}{4}e^{t} - 3te^{t}\right\}$

2 (20 **pts**.) Find

$$\mathcal{L}^{-1}\left\{\frac{-5s-36}{(s+2)(s^2+9)}\right\}$$

Solution:

$$\frac{-5s-36}{(s+2)(s^2+9)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+9}$$

Multiplying both sides by $s+2$ and letting $s = -2$ yields

$$\frac{10-36}{13} = -2 = A$$

And

$$\frac{-5s-36}{(s+2)(s^2+9)} = \frac{-2}{s+2} + \frac{Bs+C}{s^2+9}$$

Letting $s = 0$ yields

$$\frac{-36}{18} = -2 = -1 + \frac{C}{9}$$

Thus $C = -9$ and we have

$$\frac{-5s-36}{(s+2)(s^2+9)} = \frac{-2}{s+2} + \frac{Bs-9}{s^2+9}$$

Letting $s = 1$ yields

$$\frac{-5-36}{3(10)} = \frac{-2}{3} + \frac{B-9}{10}$$

Multiplying by 30 yields

$$-41 = -20 + 3B - 27$$

or $3B = 6$ so $B = 2$.

Thus

$$\frac{-5s-36}{(s+2)(s^2+9)} = \frac{-2}{s+2} + \frac{2s-9}{s^2+9}$$

$$\mathcal{L}^{-1}\left\{\frac{-5s-36}{(s+2)(s^2+9)}\right\} = -2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - 3\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$
$$= -2e^{-2t} + 2\cos 3t - 3\sin 3t$$

3 (30 **pts**.) Find the series solution near $x = 0$ of the equation

$$(x^2-1)y'' + y = 0$$

Be sure to give the recurrence relation. Indicate the two linearly independent solutions and give the first *five* nonzero terms of the solution.

Solution:

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$$y = \sum_{n=0}^{\infty} a_n x^n$$
$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

Substituting into the DE leads to

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n - \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Combining the first and third series we have

$$a_0 + a_1 x + \sum_{n=2}^{\infty} a_n[n(n-1)+1]x^n - \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} = 0$$

Let $n-2 = k$ or $n = k+2$ in the second series. Then we have

$$a_0 + a_1 x + \sum_{n=2}^{\infty} a_n[n(n-1)+1]x^n - \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k = 0$$

Replacing n and k by m we have

$$a_0 + a_1 x - 2a_2 - 3(2)a_3 x + \sum_{m=2}^{\infty}\{a_m[m(m-1)+1] - a_{m+2}(m+2)(m+1)\}x^m = 0$$

Thus

$$a_2 = \frac{1}{2}a_0$$
$$a_3 = \frac{1}{6}a_1$$

and the recurrence relation is

$$a_m(m^2 - m + 1) - a_{m+2}(m+2)(m+1) = 0 \quad m = 2, 3, \ldots$$

or

$$a_{m+2} = \frac{m^2 - m + 1}{(m+2)(m+1)}a_m \quad m = 2, 3, \ldots$$

Hence

$$a_4 = \frac{3}{4(3)}a_2 = \frac{1}{4}a_2 = \frac{1}{8}a_0$$
$$a_5 = \frac{7}{5(4)}a_3 = \frac{7}{5(4)(6)}a_1$$

Thus

$$y = a_0\left(1 + \frac{x^2}{2} + \frac{1}{8}x^4 + \cdots\right) + a_1\left(x + \frac{1}{6}x^3 + \frac{7}{(4)(5)(6)}x^5 + \cdots\right)$$

4 (25 **pts**.) Find the eigenvalues, λ, and eigenfunctions for

$$y'' + \lambda y = 0; \quad y'(0) = 0, \quad y(1) = 0$$

Be sure to consider all values of λ.

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Solution: The characteristic equation is $r^2 + \lambda = 0$. Hence $r = \pm\sqrt{-\lambda}$. We must consider three cases; $\lambda < 0$, $\lambda = 0$,, and $\lambda > 0$.

I. $\lambda < 0$. Let $\lambda = -\alpha^2$ where $\alpha \neq 0$. Then the differential equation becomes

$$y'' - \alpha^2 y = 0$$

and has the general solution

$$y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$
$$y'(x) = \alpha(c_1 e^{\alpha x} - c_2 e^{-\alpha x}).$$

$y'(0) = 0 \Rightarrow c_1 = c_2$ so

$$y = c_1(e^{\alpha x} + e^{-\alpha x})$$

Thus

$$y'(x) = c_1\alpha(e^{\alpha x} - e^{-\alpha x})$$

$y(1) = 0 \Rightarrow$

$$c_1\alpha(e^{\alpha} - e^{-\alpha}) = 0$$

Thus $c_1 = 0$ and hence $c_2 = 0$ and the only solution for this case is $y = 0$.

II. $\lambda = 0$. The solution is $y = c_1 x + c_2$. The BCs imply $c_1 = c_2 = 0$. Again the only solution is $y = 0$.

III. $\lambda > 0$. Let $\lambda = \beta^2$ where $\beta \neq 0$. The DE becomes

$$y'' + \beta^2 y = 0$$

and has the general solution

$$y = c_1\cos\beta x + c_2\sin\beta x$$
$$y'(x) = \beta(-c_1\sin\beta x + c_2\cos\beta x).$$

The BCs imply

$$y'(0) = c_2\beta = 0$$

Thus $c_2 = 0$.

$$y(1) = c_1\cos\beta = 0$$

Hence

$$\beta = \left(\frac{2n+1}{2}\right)\pi \quad n = 0, 1, 2, \ldots$$

Hence the eigenvalues are

$$\lambda = \beta^2 = \left(\frac{2n+1}{2}\right)^2\pi^2 \quad n = 0, 1, 2, \ldots$$

and the eigenfunctions are

$$y_n(x) = c_n\cos\left(\frac{2n+1}{2}\right)\pi x \quad n = 0, 1, 2, \ldots$$

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Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$		
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	$n \geq 1$	$s > 0$
e^{at}	$\frac{1}{s-a}$		$s > a$
$\sin bt$	$\frac{b}{s^2 + b^2}$		$s > 0$
$\cos bt$	$\frac{s}{s^2 + b^2}$		$s > 0$
$e^{at} f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		