Name:_____     Lecure Section ____

**Ma 221**                                  **Exam II A  Solutions**
**13S**

**1**. (30 **pts**. **total**)   Consider the differential equation

$$y'' + 4y' + 5y = 2e^{-2t} + \cos t$$

**1 a** (6 **pts**.)   Find the homogeneous solution of this equation.

Solution:  The characteristic polynomial is

$$p(r) = r^2 + 4r + 5 = 0$$

so

$$r = \frac{-4 \pm \sqrt{16 - 4(1)(5)}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

Hence

$$y_h = c_1 e^{-2t} \sin t + c_2 e^{-2t} \cos t$$

**1 b** (20 **pts**.)   Find a particular solution of this equation.

Solution: We first find a particular solution for $2e^{-2t}$. Since $p(-2) = 1 \ne 0$ so

$$y_{p_1} = \frac{2e^{-2t}}{1} = 2e^{-2t}$$

To find a particular solution for $\cos t$ we consider the two equations

$$y'' + 4y' + 5y = \cos t$$
$$v'' + 4v' + 5v = \sin t$$

Multiplying the second equation by $i$ and letting $w = y + iv$ we have

$$w'' + 4w' + 5w = \cos t + i \sin t = e^{it}$$

Since $p(i) = 4 + 4i$ then

$$w_p = \frac{e^{it}}{4 + 4i} = \frac{e^{it}}{4 + 4i} \left( \frac{4 - 4i}{4 - 4i} \right) = \frac{(4 - 4i)}{32} (\cos t + i \sin t)$$
$$= \frac{1}{8}(1 - i)(\cos t + i \sin t) = \frac{1}{8} \cos t + \frac{1}{8} \sin t + \frac{1}{8} i \sin t - \frac{1}{8} i \cos t$$

Since $y_{p_2}$ is the real part of $w_p$ then

$$y_{p_2} = \frac{1}{8} \cos t + \frac{1}{8} \sin t$$

and

$$y_p = y_{p_1} + y_{p_2} = 2e^{-2t} + \frac{1}{8} \cos t + \frac{1}{8} \sin t$$

Alternatively we can assume

$$y_{p_2} = A \cos t + B \sin t$$

so

$$y' = -A \sin t + B \cos t$$
$$y'' = -A \cos t - B \sin t$$

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Plugging into the DE we have
$$-A\cos t - B\sin t - 4A\sin t + 4B\cos t + 5A\cos t + 5B\sin t = \cos t$$

Hence
$$-B - 4A + 5B = 0$$
$$-A + 4B + 5A = 1$$

From the first equation we have that $A = B$ and the second equation tells us that $8A = 1$. Therefore $A = B = \frac{1}{8}$ and again
$$y_{p2} = \frac{1}{8}\cos t + \frac{1}{8}\sin t$$

**1 c (4 pts.)**  Give a general solution of this equation.

$$y_g = y_h + y_{p_1} + y_{p_2} = c_1 e^{-2t}\sin t + c_2 e^{-2t}\cos t + 2e^{-2t} + \frac{1}{8}\cos t + \frac{1}{8}\sin t$$

SNB check $y'' + 4y' + 5y = 2e^{-2t} + \cos t$, Exact solution is:
$$\left\{\frac{1}{8e^{2t}}\left((\cos t)e^{2t} + e^{2t}\sin t + 16\right) + C_2(\cos t)e^{-2t} - C_3(\sin t)e^{-2t}\right\}$$

**2 (20 pts. total)**  Find a particular solution of the differential equation
$$y'' - 2y' + 2y = 10t^2$$

Solution: Let
$$y_p = At^2 + Bt + C$$

Then
$$y_p' = 2At + B$$
$$y_p'' = 2A$$

Plugging into the DE we have
$$2A - 4At - 2B + 2At^2 + 2Bt + 2C = 10t^2$$

Hence $A = 5$,
$$-4A + 2B = 0$$
$$10 - 2B + 2C = 0$$

so $B = 2A = 10$ and $C = B - 5 = 5$. Thus
$$y_p = 5t^2 + 10t + 5$$

SNB check: $y'' - 2y' + 2y = 10t^2$, Exact solution is: $\left\{10t + 5t^2 + C_2(\cos t)e^t - C_3 e^t \sin t + 5\right\}$

**3 (50 pts. total)**

**3a $\left(10\ \textbf{pts.}\right)$**  One solution of the homogeneous equation
$$t^2 y'' - ty' + y = 0 \quad t > 0$$

is $y_1(t) = t$.  Find a second linearly independent solution of this equation by letting

$$y_2(t) = u(t)t$$

and determining $u(t)$.
Solution:

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$$y_2(t) = u(t)t$$
$$y_2'(t) = u + u't$$
$$y_2''(t) = u''t + 2u'$$

Substituting into the DE we have

$$u''t^3 + 2u't^2 - ut - u't^2 + ut = 0$$

or

$$u''t + u' = 0$$

This can be rewritten as

$$\left(u't\right)' = 0$$

so

$$u't = c_1$$

and

$$u = c_1 \ln t + c_2$$

Hence

$$y_2 = u(t)t = c_1 t \ln t + c_2 t$$

The second linearly independent solution is therefore $t \ln t$.

Since the Wronskian of these two functions is

$$W[t, t\ln t] = \begin{vmatrix} t & t\ln t \\ 1 & \ln t + 1 \end{vmatrix} = t \neq 0 \text{ for } t > 0$$

the two solutions are linearly independent.

**3b** $\left(5 \text{ pts.}\right)$   Give a general homogenous solution to the equation

$$t^2 y'' - ty' + y = 0 \quad t > 0$$

Solution:

$$y_h = c_1 t \ln t + c_2 t$$

**3c** $\left(25 \text{ pts.}\right)$   Given that $y_1(t) = t$ and $y_2(t) = t\ln t$ are two linearly independent solutions of the homogeneous equation

$$t^2 y'' - ty' + y = 0$$

find a particular solution to the equation

$$t^2 y'' - ty' + y = t$$

Solution: We use the Method of Variation of Parameters. Let

$$y_p = v_1(t)t + v_2(t)t\ln t$$

Then the equations for $v_1'$ and $v_2'$ are

$$v_1't + v_2't\ln t = 0$$

$$v_1' + v_2'(\ln t + 1) = \frac{t}{t^2} = \frac{1}{t}$$

Multiplying the second equation by $t$ and subtracting it from the first yields

$$-tv_2' = -1$$

so

$$v_2 = \int \frac{1}{t}dt = \ln t$$

Also from the first equation

$$v_1't + \ln t = 0$$

so

$$v_1 = -\int \frac{\ln t}{t}dt = -\frac{(\ln t)^2}{2}$$

Therefore

$$y_p = -t\frac{(\ln t)^2}{2} + (t\ln t)^2 = t\frac{(\ln t)^2}{2}$$

**3d** $\left(\textbf{10 pts.}\right)$   Solve the initial value problem

$$t^2 y'' - ty' + y = t \quad y(1) = 1, \quad y'(1) = 4$$

Solution:

$$y = y_h + y_p = c_1 t\ln t + c_2 t + t\frac{(\ln t)^2}{2}$$

$$y' = c_1 \ln t + c_1 + c_2 + \frac{(\ln t)^2}{2} + \frac{\ln t}{t}$$

$$y(1) = c_2 = 1$$

$$y'(1) = c_1 + 1 = 4$$

so $c_1 = 3$ and

$$y = 3t\ln t + t + t\frac{(\ln t)^2}{2} \quad t > 0$$

SNB check

$$t^2 y'' - ty' + y = t$$

$$y(1) = 1$$

$$y'(1) = 4$$

, Exact solution is: $\left\{ \frac{1}{2}t\ln^2 t + 3t\ln t + t \right\}$

# Table of Integrals

$$\int \ln t\, dt = t(\ln t - 1) + C$$

$$\int (\ln t)^2 dt = t\left(\ln^2 t - 2\ln t + 2\right) + C$$

$$\int \frac{\ln t}{t} dt = \frac{1}{2}\ln^2 t + C$$

$$\int \frac{(\ln t)^2}{t} dt + C = \frac{1}{3}\ln^3 t + C$$

$$\int \frac{1}{t\ln t} dt = \ln(\ln t) + C$$