Ma 221 **Exam II B Solutions**
13S

1. (30 **pts**. **total**) Consider the differential equation

$$y'' - 4y' + 5y = 2e^{2t} + \sin t$$

1 a (6 **pts**.) Find the homogeneous solution of this equation.

Solution: The characteristic polynomial is

$$p(r) = r^2 - 4r + 5 = 0$$

so

$$r = \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

Hence

$$y_h = c_1 e^{2t} \sin t + c_2 e^{2t} \cos t$$

1 b (20 **pts**.) Find a particular solution of this equation.

Solution: We first find a particular solution for $2e^{2t}$. Since $p(2) = 1 \neq 0$ so

$$y_{p_1} = \frac{2e^{2t}}{1} = 2e^{2t}$$

To find a particular solution for $\sin t$ we consider the two equations

$$x'' - 4x' + 5x = \cos t$$
$$y'' - 4y' + 5y = \sin t$$

Multiplying the second equation by i and letting $z = x + iy$ we have

$$z'' + 4z' + 5z = \cos t + i \sin t = e^{it}$$

Since $p(i) = 4 - 4i$ then

$$z_p = \frac{e^{it}}{4 - 4i} = \frac{e^{it}}{4 - 4i}\left(\frac{4 + 4i}{4 + 4i}\right) = \frac{(4 + 4i)}{32}(\cos t + i \sin t)$$
$$= \frac{1}{8}(1 + i)(\cos t + i \sin t) = \frac{1}{8}\cos t - \frac{1}{8}\sin t + \frac{1}{8} i \sin t + \frac{1}{8} i \cos t$$

Since y_{p_2} is the imaginary part of z_p then

$$y_{p_2} = \frac{1}{8}\cos t + \frac{1}{8}\sin t$$

and

$$y_p = y_{p_1} + y_{p_2} = 2e^{2t} + \frac{1}{8}\cos t + \frac{1}{8}\sin t$$

Alternatively we can assume

$$y_{p_2} = A\cos t + B\sin t$$

so

$$y' = -A\sin t + B\cos t$$
$$y'' = -A\cos t - B\sin t$$

Plugging into the DE we have

$$-A\cos t - B\sin t + 4A\sin t - 4B\cos t + 5A\cos t + 5B\sin t = \sin t$$

Hence

$$-B + 4A + 5B = 1$$
$$-A - 4B + 5A = 0$$

From the fsecond equation we have that $A = B$ and the first equation then tells us that $8A = 1$. Therefore $A = B = \frac{1}{8}$ and again

$$y_{p_2} = \frac{1}{8}\cos t + \frac{1}{8}\sin t$$

1 c (4 **pts**.) Give a general solution of this equation.

$$y_g = y_h + y_{p_1} + y_{p_2} = c_1 e^{2t}\sin t + c_2 e^{2t}\cos t + 2e^{2t} + \frac{1}{8}\cos t + \frac{1}{8}\sin t$$

SNB check $y'' - 4y' + 5y = 2e^{2t} + \sin t$, Exact solution is:
$\left\{2e^{2t} + \frac{1}{8}\cos t + \frac{1}{8}\sin t + C_{11}(\cos t)e^{2t} - C_{12}(\sin t)e^{2t}\right\}$

2 (20 **pts**. **total**) Find a particular solution of the differential equation

$$y'' + 2y' + 2y = 8t^2$$

Solution: Let

$$y_p = At^2 + Bt + C$$

Then

$$y_p' = 2At + B$$
$$y_p'' = 2A$$

Plugging into the DE we have

$$2A + 4At + 2B + 2At^2 + 2Bt + 2C = 8t^2$$

Hence $A = 4$,

$$4A + 2B = 0$$
$$2A + 2B + 2C = 0$$

so $B = -2A = -8$ and $C = -A - B = 4.$ Thus

$$y_p = 4t^2 - 8t + 4$$

SNB check: $y'' + 2y' + 2y = 8t^2$, Exact solution is: $\left\{4t^2 - 8t + C_2(\cos t)e^{-t} - C_3(\sin t)e^{-t} + 4\right\}$,

3 (**50 pts**. **total**)

3a (10 **pts**.) One solution of the homogeneous equation

$$t^2y'' - ty' + y = 0 \quad t > 0$$

is $y_1(t) = t.$ Find a second linearly independent solution of this equation by letting

$$y_2(t) = u(t)t$$

and determining $u(t)$.

Solution:

$$y_2(t) = u(t)t$$
$$y_2'(t) = u + u't$$
$$y_2''(t) = u''t + 2u'$$

Substituting into the DE we have

$$u''t^3 + 2u't^2 - ut - u't^2 + ut = 0$$

or

$$u''t + u' = 0$$

This can be rewritten as

$$\left(u't\right)' = 0$$

so

$$u't = c_1$$

and

$$u = c_1 \ln t + c_2$$

Hence

$$y_2 = u(t)t = c_1 t \ln t + c_2 t$$

The second linearly independent solution is therefore $t \ln t$.

Since the Wronskian of these two functions is

$$W[t, t\ln t] = \begin{vmatrix} t & t\ln t \\ 1 & \ln t + 1 \end{vmatrix} = t \neq 0 \text{ for } t > 0$$

the two solutions are linearly independent.

3b (5 **pts**.) Give a general homogenous solution to the equation

$$t^2y'' - ty' + y = 0 \quad t > 0$$

Solution:

$$y_h = c_1 t \ln t + c_2 t$$

3c (25 **pts**.) Given that $y_1(t) = t$ and $y_2(t) = t\ln t$ are two linearly independent solutions of the homogeneous equation

$$t^2y'' - ty' + y = 0$$

find a particular solution to the equation

$$t^2y'' - ty' + y = t$$

Solution: We use the Method of Variation of Parameters. Let

$$y_p = v_1(t)t + v_2(t)t\ln t$$

Then the equations for v_1' and v_2' are

$$v_1' t + v_2' t \ln t = 0$$
$$v_1' + v_2'(\ln t + 1) = \frac{t}{t^2} = \frac{1}{t}$$

Multiplying the second equation by t and subtracting it from the first yields

$$-tv_2' = -1$$

so

$$v_2 = \int \frac{1}{t} dt = \ln t$$

Also from the first equation

$$v_1' t + \ln t = 0$$

so

$$v_1 = -\int \frac{\ln t}{t} dt = -\frac{(\ln t)^2}{2}$$

Therefore

$$y_p = -t\frac{(\ln t)^2}{2} + (t \ln t)^2 = t\frac{(\ln t)^2}{2}$$

3c (10 **pts**.) Solve the initial value problem

$$t^2 y'' - ty' + y = t \quad y(1) = 1, \quad y'(1) = 4$$

Solution:

$$y = y_h + y_p = c_1 t \ln t + c_2 t + t\frac{(\ln t)^2}{2}$$

$$y' = c_1 \ln t + c_1 + c_2 + \frac{(\ln t)^2}{2} + \frac{\ln t}{t}$$

$$y(1) = c_2 = 1$$
$$y'(1) = c_1 + 1 = 4$$

so $c_1 = 3$ and

$$y = t \ln t + 3t + t\frac{(\ln t)^2}{2} \quad t > 0$$

SNB check

$$t^2 y'' - ty' + y = t$$
$$y(1) = 1$$
$$y'(1) = 4$$

, Exact solution is: $\left\{\frac{1}{2} t \ln^2 t + 3t \ln t + t\right\}$

Table of Integrals

$$\int \ln t\,dt = t(\ln t - 1) + C$$

$$\int (\ln t)^2 dt = t\left(\ln^2 t - 2\ln t + 2\right) + C$$

$$\int \frac{\ln t}{t} dt = \frac{1}{2}\ln^2 t + C$$

$$\int \frac{(\ln t)^2}{t} dt + C = \frac{1}{3}\ln^3 t + C$$

$$\int \frac{1}{t\ln t} dt = \ln(\ln t) + C$$