Name:__________________________________ Lecure Section ___

Ma 221 Exam IIIA Solutions 13S

1a (10 **pts**.) Let $\hat{f}(s) = \mathcal{L}\{f(t)\}$. Show that

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+\alpha) = \hat{f}(s+a)$$

Solution:

$$\mathcal{L}\{e^{-at}f(t)\} = \int_0^\infty e^{-st}e^{-at}f(t)dt = \int_0^\infty e^{-(s+a)t}f(t)dt = F(s+\alpha) = \hat{f}(s+a)$$

1b (15 **pts**.) Find

$$\mathcal{L}^{-1}\left(\frac{1}{s^3+s}\right)$$

Solution:

$$\frac{1}{s^3+s} = \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

Multiplying by s and setting $s = 0$ gives $A = 1$. Thus we have

$$\frac{1}{s^3+s} = \frac{1}{s(s^2+1)} = \frac{1}{s} + \frac{Bs+C}{s^2+1}$$

Setting $s = 1$ gives

$$\frac{1}{2} = 1 + \frac{B+C}{2}$$

Setting $s = -1$ gives

$$-\frac{1}{2} = -1 + \frac{-B+C}{2}$$

or

$$\frac{B+C}{2} = -\frac{1}{2}$$

and

$$\frac{-B+C}{2} = \frac{1}{2}$$

Adding the two equations yields $C = 0$ and thus $B = -1$. Therefore

$$\frac{1}{s^3+s} = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

so that

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{1}{s^3+s}\right) &= \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)\\ &= 1-\cos t\end{aligned}$$

2 (25 **pts**.) Use Laplace Transforms to solve the initial value problem

$$y'' + y' - 2y = 4 \quad y(0) = 2 \;\; y'(0) = 1$$

Solution: Taking Laplace Transforms of both sides we have

Name:________________________________ Lecure Section ___

$$(s^2+s-2)\mathcal{L}\{y\} - sy(0) - y'(0) - y(0) = \frac{4}{s}$$

or after substituting in the initial conditions

$$(s^2+s-2)\mathcal{L}\{y\} - 2s - 1 - 2 = \frac{4}{s}$$

Therefore

$$(s^2+s-2)\mathcal{L}\{y\} = \frac{4}{s} + 2s + 3 = \frac{2s^2+3s+4}{s}$$

Hence

$$\mathcal{L}\{y\} = \frac{2s^2+3s+4}{s(s-1)(s+2)}$$

Now

$$\frac{2s^2+3s+4}{s(s-1)(s+2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+2}$$

Multiplying by s and setting $s = 0$ yields $A = \frac{4}{-2} = -2$. Similarly $B = \frac{2+3+4}{3} = 3$ and $C = \frac{8-6+4}{(-2)(-3)} = 1$. Thus

$$y = \mathcal{L}^{-1}\left(\frac{2s^2+3s+4}{s(s-1)(s+2)}\right) = \mathcal{L}^{-1}\left(\frac{-2}{s} + \frac{3}{s-1} + \frac{1}{s+2}\right)$$
$$= -2 + 3e^t + e^{-2t}$$

3 (25 **pts**.) Find the first 5 nonzero terms of the power series solution about $x = 0$ for the DE:

$$y'' - 2xy' + 2y = 0$$

Be sure to give the recurrence relation.
Solution:

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

so

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1)x^{n-2}$$

.
The differential equation ⇒

$$\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} - 2\sum_{n=1}^{\infty} a_n n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

Combining the second and third summations we have

$$\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} + 2a_0 + 2\sum_{n=1}^{\infty} a_n(1-n)x^n = 0$$

Shifting the first series by letting $n-2 = k$ or $n = k+2$ we have

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k + 2a_0 + 2\sum_{n=1}^{\infty} a_n(1-n)x^n = 0$$

or after replacing k and n by m and combining the series

$$2a_0 + 2a_2 + \sum_{m=1}^{\infty}[a_{m+2}(m+2)(m+1) + 2a_m(1-m)]x^m = 0$$

Thus

$$a_2 = -a_0$$

and the recurrence relation is

$$a_{m+2} = \frac{2(m-1)}{(m+2)(m+1)}a_m \quad m = 1,2,3,\ldots$$

Thus

$$a_3 = 0$$

and hence

$$a_{2n+1} = 0 \text{ for } n = 1,2,3,\ldots$$

Also

$$a_4 = \frac{2}{4(3)}a_2 = \frac{1}{6}a_2 = -\frac{1}{6}a_0$$

$$a_6 = \frac{2(3)}{6(5)}a_4 = \frac{1}{5}a_4 = -\frac{1}{30}a_0$$

Thus

$$y = a_1x + a_0\left(1 - x^2 - \frac{1}{6}x^4 - \frac{1}{30}x^6 - \cdots\right)$$

SNB check: $y'' - 2xy' + 2y = 0$, Series solution is: $\left\{y(0) + xy'(0) - x^2y(0) - \frac{1}{6}x^4y(0) - \frac{1}{30}x^6y(0) + O(x^7)\right\}$,

4 $\left(25\ \textbf{pts.}\right)$ Find all eigenvalues (λ) and the corresponding eigenfunctions for the boundary value problem

$$y'' - 2y + \lambda y = 0 \quad y'(0) = y'(\pi) = 0$$

Solution: The characteristic equation of the d. e. is

$$r^2 - 2 + \lambda = 0$$
$$r^2 = 2 - \lambda$$
$$r = \pm\sqrt{2-\lambda}$$

There are three cases to be considered depending on whether the quantity under the radical is positive, zero or negative. We deal with each case in turn.
Case I: $2 - \lambda > 0$.. Let's write $\mu^2 = 2 - \lambda$. Hence $r = \pm\mu$ and the solution to the d.e. is

$$y = c_1e^{\mu x} + c_2e^{-\mu x}$$
$$y' = \mu(c_1e^{\mu x} - c_2e^{-\mu x}).$$

From $y'(0) = 0$,

$$c_1 = c_2.$$

From $y'(\pi) = 0$,

$$\mu c_1(e^{\mu\pi} + e^{\mu\pi}) = 0.$$

Hence $c_1 = c_2 = 0$ and there is no non-zero solution.

Case II: $2 - \lambda = 0$. The d.e. is $y'' = 0$. The solution is

$$y = c_1 x + c_2$$
$$y' = c_1$$

From $y'(0) = 0$,

$$c_1 = 0.$$

Then $y'(\pi) = 0$ and c_2 is arbitrary. Hence we have an eigenvalue of $\lambda = 2$ and will label this as $\lambda_0 = 2$ with the corresponding eigenfunction labeled as $y_0 = c_0$.

Case III. $2 - \lambda < 0$. Let's write $2 - \lambda = -\mu^2$. Hence $r = \pm\sqrt{2-\lambda} = \pm\mu i$ and the solution to the d.e. is

$$y = c_1 \cos \mu x + c_2 \sin \mu x$$
$$y' = \mu(-c_1 \sin \mu x + c_2 \cos \mu x)$$

From $y'(0) = 0$,

$$c_2 = 0.$$

From $y'(\pi) = 0$,

$$-\mu c_1 \sin \mu\pi = 0.$$

For a non-zero solution, we need to have $\sin \mu\pi = 0$. So eigenvalues and eigenfunctions come from

$$\begin{aligned} \mu_n &= n \qquad && n = 1,2,3,\ldots \\ \lambda_n &= 2 + \mu^2 && \\ &= 2 + n^2 && n = 1,2,3,\ldots \\ y_n &= c_n \cos nx && n = 1,2,3,\ldots \end{aligned}$$

For the ease of the grader, we can combine cases II and III and summarize the results as

$$\text{eigenvalues}: \ \lambda_n = 2 + n^2 \qquad n = 0,1,2,3,\ldots$$
$$\text{eigenfunctions}: \ y_n = c_n \cos nx \qquad n = 0,1,2,3,..$$

Name:____________________________________ Lecure Section ___

Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s) = \hat{f}(s)$		
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	$n \geq 1$	$s > 0$
e^{at}	$\frac{1}{s-a}$		$s > a$
$\sin bt$	$\frac{b}{s^2 + b^2}$		$s > 0$
$\cos bt$	$\frac{s}{s^2 + b^2}$		$s > 0$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		