Ma 221   Exam IIIB Solutions
13S

1a $\left(\textbf{10 pts.}\right)$ Let $\widehat{f}(s) = \mathcal{L}\{f(t)\}$. Show that

$$\mathcal{L}\{e^{-at}f(t)\} = F(s + \alpha) = \widehat{f}(s + a)$$

Solution:

$$\mathcal{L}\{e^{-at}f(t)\} = \int_0^\infty e^{-st}e^{-at}f(t)dt = \int_0^\infty e^{-(s+a)t}f(t)dt = F(s + \alpha) = \widehat{f}(s + a)$$

1b $\left(\textbf{15 pts.}\right)$ Find

$$\mathcal{L}^{-1}\left(\frac{4}{s^3 + 4s}\right)$$

Solution:

$$\frac{4}{s^3 + 4s} = \frac{4}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4}$$

Multiplying by the common denominator, $s^3 + 4s$, gives

$$4 = A(s^2 + 4) + (Bs + C)s. \tag{1}$$

Setting $s = 0$ gives

$$4 = 4A.$$

Thus $A = 1$. Equating the coefficient of s^2 on each side gives

$$0 = A + B.$$

$B = -A = -1$. Finally, we equate the coefficients of $s.$ in equation (1)

$$0 = C.$$

Therefore

$$\frac{4}{s^3 + 4s} = \frac{4}{s(s^2 + 4)} = \frac{1}{s} - \frac{s}{s^2 + 4}$$

so that

$$\mathcal{L}^{-1}\left(\frac{4}{s^3 + 4s}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{s}{s^2 + 4}\right)$$

$$= 1 - \cos 2t$$

2 $\left(25\ \textbf{pts.}\right)$ Use Laplace Transforms to solve the initial value problem

$$y'' - y' - 2y = 4 \quad y(0) = 2 \ \ y'(0) = 5$$

Solution: Taking Laplace Transforms of both sides we have

$$(s^2 - s - 2)\mathcal{L}\{y\} - sy(0) - y'(0) + y(0) = \frac{4}{s}$$

or after substituting in the initial conditions

$$(s^2 - s - 2)\mathcal{L}\{y\} - 2s - 5 + 2 = \frac{4}{s}$$

Therefore

$$(s^2 - s - 2)\mathcal{L}\{y\} = \frac{4}{s} + 2s + 3 = \frac{2s^2 + 3s + 4}{s}$$

Hence

$$\mathcal{L}\{y\} = \frac{2s^2 + 3s + 4}{s(s+1)(s-2)}$$

Now

$$\frac{2s^2 + 3s + 4}{s(s+1)(s-2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-2}$$

Multiplying by s and setting $s = 0$ yields $A = \frac{4}{-2} = -2$. Similarly $B = \frac{2-3+4}{(-1)(-3)} = 1$ and $C = \frac{8+6+4}{(2)(3)} = 3$. Thus

$$y = \mathcal{L}^{-1}\left(\frac{2s^2 + 3s + 4}{s(s+1)(s-2)}\right) = \mathcal{L}^{-1}\left(\frac{-2}{s} + \frac{1}{s+1} + \frac{3}{s-2}\right)$$

$$= -2 + e^t + 3e^{2t}$$

3 $\left(\textbf{25 pts.}\right)$ Find the first 5 nonzero terms of the power series solution about $x = 0$ for the DE:

$$y'' + 2xy' - 2y = 0$$

Be sure to give the recurrence relation.
Solution:

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

so

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2}$$

.

The differential equation $\Rightarrow$

$$\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} + 2\sum_{n=1}^{\infty} a_n n x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

Combining the second and third summations we have

$$\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} - 2a_0 + 2\sum_{n=1}^{\infty} a_n(n-1)x^n = 0$$

Shifting the first series by letting $n - 2 = k$ or $n = k + 2$ we have

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k - 2a_0 + 2\sum_{n=1}^{\infty} a_n(n-1)x^n = 0$$

or after replacing k and n by m and combining the series

$$2a2 - 2a_0 + \sum_{m=1}^{\infty}[a_{m+2}(m+2)(m+1) + 2a_m(m-1)]x^m = 0$$

Thus

$$a_2 = a_0$$

and the recurrence relation is

$$a_{m+2} = \frac{2(1-m)}{(m+2)(m+1)} a_m \quad m = 1, 2, 3, \ldots$$

Thus

$$a_3 = 0$$

and hence

$$a_{2n+1} = 0 \text{ for } n = 1, 2, 3, \ldots$$

Also

$$a_4 = \frac{-2}{4(3)} a_2 = \frac{-1}{6} a_2 = -\frac{1}{6} a_0$$

$$a_6 = \frac{2(-3)}{6(5)} a_4 = \frac{-1}{5} a_4 = \frac{1}{30} a_0$$

Thus

$$y = a_1 x + a_0 \left(1 + x^2 - \frac{1}{6} x^4 + \frac{1}{30} x^6 - \cdots \right)$$

SNB check: $y'' + 2xy' - 2y = 0$, Series solution is: $\left\{ y(0) + xy'(0) + x^2 y(0) - \frac{1}{6} x^4 y(0) + O(x^5) \right\}$,

4 $\left(\textbf{25 pts.} \right)$ Find all eigenvalues (λ) and the corresponding eigenfunctions for the boundary value problem

$$y'' - 3y + \lambda y = 0 \quad y'(0) = y'(\pi) = 0$$

Solution: The characteristic equation of the d. e. is

$$r^2 - 3 + \lambda = 0$$
$$r^2 = 3 - \lambda$$
$$r = \pm \sqrt{3 - \lambda}$$

There are three cases to be considered depending on whether the quantity under the radical is positive, zero or negative. We deal with each case in turn.

Case I: $3 - \lambda > 0$.. Let's write $\mu^2 = 3 - \lambda$. Hence $r = \pm \mu$ and the solution to the d.e. is

$$y = c_1 e^{\mu x} + c_2 e^{-\mu x}$$
$$y' = \mu(c_1 e^{\mu x} - c_2 e^{-\mu x}).$$

From $y'(0) = 0$,

$$c_1 = c_2.$$

From $y'(\pi) = 0$,

$$\mu c_1 (e^{\mu \pi} + e^{\mu \pi}) = 0.$$

Hence $c_1 = c_2 = 0$ and there is no non-zero solution.

Case II: $3 - \lambda = 0$. The d.e. is $y'' = 0$. The solution is

$$y = c_1 x + c_2$$
$$y' = c_1$$

From $y'(0) = 0$,

$$c_1 = 0.$$

Then $y'(\pi) = 0$ and c_2 is arbitrary. Hence we have an eigenvalue of $\lambda = 3$ and will label this as $\lambda_0 = 3$ with the corresponding eigenfunction labeled as $y_0 = c_0$.

Case III. $3 - \lambda < 0$. Let's write $3 - \lambda = -\mu^2$. Hence $r = \pm\sqrt{3-\lambda} = \pm\mu i$ and the solution to the d.e. is
$$y = c_1 \cos\mu x + c_2 \sin\mu x$$
$$y' = \mu(-c_1 \sin\mu x + c_2 \cos\mu x)$$

From $y'(0) = 0$,
$$c_2 = 0.$$

From $y'(\pi) = 0$,
$$-\mu c_1 \sin\mu\pi = 0.$$

For a non-zero solution, we need to have $\sin\mu\pi = 0$. So eigenvalues and eigenfunctions come from
$$\mu_n = n \qquad n = 1, 2, 3, \ldots$$
$$\lambda_n = 3 + \mu^2$$
$$= 3 + n^2 \qquad n = 1, 2, 3, \ldots$$
$$y_n = c_n \cos nx \qquad n = 1, 2, 3, \ldots$$

For the ease of the grader, we can combine cases II and III and summarize the results as
$$\text{eigenvalues}: \quad \lambda_n = 3 + n^2 \qquad n = 0, 1, 2, 3, \ldots$$
$$\text{eigenfunctions}: \quad y_n = c_n \cos nx \qquad n = 0, 1, 2, 3, \ldots$$

Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s) = \widehat{f}(s)$		
$\dfrac{t^{n-1}}{(n-1)!}$	$\dfrac{1}{s^{n}}$	$n \geq 1$	$s > 0$
e^{at}	$\dfrac{1}{s-a}$		$s > a$
$\sin bt$	$\dfrac{b}{s^2 + b^2}$		$s > 0$
$\cos bt$	$\dfrac{s}{s^2 + b^2}$		$s > 0$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		