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Ma 221**Exam IA Solutions****14S**

Solve the following equations:

1 [25 pts.]

$$(3x^2 - 2xy + 2)dx + (6y^2 - x^2 + 3)dy = 0$$

Solution: If we let $M = 3x^2 - 2xy + 2$ and $N = 6y^2 - x^2 + 3$, then

$$M_y = -2x = N_x$$

Hence this equation is exact. Thus there exist $f(x, y)$ such that

$$f_x = M = 3x^2 - 2xy + 2 \quad \text{and} \quad f_y = N = 6y^2 - x^2 + 3$$

Integrating f_x with respect to x leads to

$$f = x^3 - x^2y + 2x + g(y)$$

Therefore

$$f_y = -x^2 + g'(y) = N = 6y^2 - x^2 + 3$$

We see that

$$g'(y) = 6y^2 + 3$$

so

$$g(y) = 2y^3 + 3y + C$$

Hence

$$f = x^3 - x^2y + 2x + 2y^3 + 3y + C$$

and the solution is given by

$$f = x^3 - x^2y + 2x + 2y^3 + 3y = K$$

2 [25 pts.]

$$\frac{dy}{dx} = \frac{y \cos x}{1 + 2y^2}; \quad y(0) = 1$$

Solution: This equation is separable, since it may be rewritten as

$$\cos x dx - \left(\frac{1 + 2y^2}{y} \right) dy = 0$$

or

$$\cos x dx - \left(\frac{1}{y} + 2y \right) dy = 0$$

Integrating we have

$$\sin x - \ln|y| - y^2 = C$$

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The initial condition implies

$$-\ln 1 - (1)^2 = C$$

so $C = -1$ and the solution is

$$\sin x - \ln y - y^2 = -1$$

3 [25 points]

$$\cos^2 t \sin ty' = -\cos^3 ty + 1 \quad y\left(\frac{\pi}{4}\right) = 0$$

Solution: We may rewrite the equation as

$$y' + \frac{\cos t}{\sin t} y = \frac{1}{\cos^2 t \sin t}$$

This is a first order linear DE. The integrating factor is

$$e^{\int P(t)dt} = e^{\int \frac{\cos t}{\sin t} dt} = e^{\ln|\sin t|} = \sin t$$

Multiplying the DE by $\sin t$ we have

$$\sin ty' + \cos ty = \frac{1}{\cos^2 t}$$

or

$$\frac{d((\sin t)y)}{dt} = \frac{1}{\cos^2 t} = \sec^2 t$$

$$\sin ty = \tan t + c$$

The initial condition $y\left(\frac{\pi}{4}\right) = 0$ implies

$$0 = 1 + c$$

so $c = -1$ and the solution is given by

$$\sin ty = \tan t - 1$$

4 [25 pts.]

$$\frac{dy}{dx} + \frac{1}{3}y = e^x y^4$$

Solution: This is a Bernoulli equation. We rewrite it as

$$y^{-4} \frac{dy}{dx} + \frac{1}{3}y^{-3} = e^x$$

Let $z = y^{-3}$. Then $z' = -3y^{-4}y'$ and we may write the above DE as

$$\frac{z'}{-3} + \frac{1}{3}z = e^x$$

or

$$z' - z = -3e^x$$

This is first order linear in z . The integrating factor is

$$e^{\int P dx} = e^{-\int dx} = e^{-x}$$

Multiplying the DE by e^{-x} we have

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$$e^{-x}z' - e^{-x}z = -3$$

or

$$\frac{d(e^{-x}z)}{dx} = -3$$

Integrating we have

$$e^{-x}z = -3x + c$$

Thus

$$z = y^{-3} = -3xe^x + ce^x$$

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Table of Integrals

$\int \sec^2 t dt = \tan t + C$
$\int \tan t dt = \ln(\sec t) + C$
$\int \frac{\sec^2 t}{\tan t} dt = \ln(\tan t) + C$
$\int t e^{at} dt = \frac{1}{a^2} e^{at} (at - 1) + C$
$\int t^2 e^{at} dt = \frac{1}{a^3} e^{at} (a^2 t^2 - 2at + 2) + C$
$\int \cos^2 t dt = \frac{1}{2} t + \frac{1}{4} \sin 2t + C$
$\int \cos^3 t dt = \frac{1}{3} \cos^2 t \sin t + \frac{2}{3} \sin t + C$
$\int \sin^2 t dt = \frac{1}{2} t - \frac{1}{4} \pi - \frac{1}{4} \sin 2t + C$
$\int \sin^3 t dt = \frac{1}{12} \cos 3t - \frac{3}{4} \cos t + C$